Differential Equations

An Introduction to Modeling Our Physical World

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First printing, Fall 2021. Second draft, Spring 2026.

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0.1 Disclaimer

These are an experiment. They won't be perfect, and they are not yet complete. The formatting will be weird sometimes, and there may be gaps, typos, or other errors in the text. I'll also add notes to myself in [red] for further editing. I welcome your feedback to improve these notes, but I may not be able to implement everything as we progress through the course. Thanks for reading these, and I'll see you in class!

0.2 Introduction

Differential Equations is a course primarily in the equations and their solutions, but the reason you are taking this course is because differential equations are used to model real-world applications. So, we will also explore how these equations relate to those models. I am an applied mathematician, so I teach this material in the context of its physical representations and the applications you might encounter as a scientist. I want to see that you understand what you are calculating, not just that you *can* calculate. These notes are meant to fill in gaps so you can solidify concepts and grasp the material. Please let me know if you have questions!

0.3 Calculus Review list

These are topics that you should be comfortable with upon entering this course. If there is anything on this list that you need help with, please SEE ME SOON. Neither of us wants you to fall behind.

1. Continuity (Calc I)
2. Chain Rule (Calc I)
3. Max/Min Problems (Calc I and III)
4. Antiderivatives (Calc I)
5. Integrals: U-Sub, Integration by Parts, Trigonometric Substitution, Trigonometric Integrals, Partial Fraction Decomposition, and basic rules of integration (Calc II) **This is the most important of these!**
6. Functions of Multiple Variables (Calc III)

7. Partial Derivatives (Calc III)
   Also helpful:
8. Taylor Series (Calc II)
9. Vector Fields (Calc III)

1. Introduction to Differential Equations andModeli

Welcome to Differential Equations! This first chapter lays a foundation for the rest of this course and then outlines techniques for solving first order differential equations. We will outline some terminology and relationships, but we will also discuss mathematical modeling and several differential equations models that we will solve later this semester. The goal of this chapter is to give you a thorough introduction to differential equations and start solving them using techniques you learned in Calculus.

1.1 Introduction to Modeling with Differential Equations

This section is an introduction to differential equations and mathematical modeling. You might be wondering, what is a differential equation? Good question. A differential equation is simply an equation that involves a derivatives of a function. We solve differential equations to determine that function. Since you have completed Calculus, you are already aware of derivatives, their relationship to the original function, and integration techniques. This course will utilize that knowledge and build upon it to give you the tools to:

- Construct differential equations to model relationcships between variables
- Solve differential equations to define a function representing the relationship between variables in an explicit form
- Evaluate the real-world behavior of a variable defined by the solution

1.1.1 Basic Knowledge for Differential Equations

First, it's important to lay some groundwork. This is a brief outline of some vocabulary and examples to show you ideas we'll be building on throughout the semester.

A differential equation is a an equation that involves derivatives. All of these are differential equations:

$\dfrac{dy}{dt} = 2y$

$y'' - 2y' + y = 7$

$x'' - x = 0$
$x' + sin(x) = 0$
$u_x + u_y = 0$
$u_{xx} + u_{yy} = 2u_x$

The last two are referred to as partial differential equations, because they involve partial derivatives ($u(x,y)$ is a function of x and y). Partial differential equations is its own course, because it is a fully rich topic that builds upon the work we do in this course. All of the listed equations are differential equations, but we use the term ordinary differential equations to describe the differential equations where the function is dependent on only one variable. This course focuses on ordinary differential equations.

Notice that the notation can use $\frac{d}{dt}$ notation (the t can also be x or other input variables), or $'$ notation to denote a derivative with respect to a single input variable. In the case of partial derivatives, we can use $\frac{\partial}{\partial x}$ notation (x can be other input variables, of course), or we can use the subscript notation like in the above examples, u_x denoting a derivative of a multivariable function u with respect to x.

Some other basic concepts and terminology of differential equations include:

- Order: The highest derivative in the equation
  only a first derivative? First order
  Has only a second derivative, or a first and second derivative? Second order.
  etc.
  Whichever derivative involved the most derivatives taken - that's the order.
  $y' = t$ first order
  $y'' - y = 0$ second order
  $y - y'''' = 2t$ fourth order
- Homogeneity: Are there terms that do not involve the function? Can you move all terms of the function (y, for example) to one side of the equation, and then it is equal to zero? The function is homogeneous if when all terms of the function are moved to one side of the equation, the other side is zero.
  $y'' - y = 0$ and $y'' = y$ are the same equation, and they are both homogeneous, the first one requires no algebra to show that it is homogeneous, the second requires a little work to show that it is homogeneous.
  $y'' = t$ is **not** homogeneous. The t is the independent variable, and is not a function of our function y. So, its presence makes the equation nonhomogeneous.
- Autonomy: An autonomous differential equation is typically a first order differential equation where the derivative is not dependent on the independent variable. $y' = f(y)$
  $y' = 2y$ is autonomous
  $y' = 2t$ is not autonomous
  $y' = 2ty$ is not autonomous
- Linearity: This is a deeper concept, but let's try to break it down. Is the equation linear in the functional terms? Are any of the terms involving y, or its derivatives, nonlinear? A derivative is a linear operator on y, so linear terms of y and its derivatives can combine in a linear differential equation. A linear function should look like it did in algebra $mx + b$, but now the x is the function y and/or any of its derivatives. (Of course, we will see many examples where the function is not y, sometimes it's x, or P, etc.) Later, we'll talk a little more about linear operators.
  $y'' - y = 0$ is linear
  $y'' - y^2 = 0$ is nonlinear. (The y is squared, so it is no longer linear)

$y'' - y = sin(t)$ is linear. (The nonlinear term is a function of t, not y, so it's still a linear DE.)

$\sin(y') = y$ is nonlinear. (The Sine function is a nonlinear function, and it is operating on the derivative of y, so the DE is nonlinear.)

In differential equations, we will sometimes have specific forms that are used to model a given relationship. A very common choice is the exponential model for populations described generally by $P' = kP$. When I read this equation it is telling me that the population, P, is changing (derivatives tell us rates of change, remember?) at a rate that is proportional to the current population, kP. In situations like this one, there are parameters, like k in this example, that are generally defined for the model used. The value of k depends on the rate of increase (or decrease) in the population modeled. So, if the population increases by 10% per time period (whatever units are used in the problem), then $k = 0.1$ because the rate of change in the population is 10% of the population. It is positive because it is increasing - we are adding 10% to the population. Many differential equations are constructed using this type of reasoning - take note.

1.1.2 Mathematical Modeling

Differential equations are one of the most commonly used mathematical models. Mathematical modeling is another course, and an entire area of research. However, when we're using differential equations, a basic understanding of mathematical modeling is helpful because differential equations are heavily used in mathematical modeling.

A model is a representation of something else: e.g. model airplane, model train, analogies, etc.

So, a mathematical model is a representation using mathematical concepts to describe a bigger problem. However, the smaller problem keeps key elements of the bigger problem so we can **learn** about the bigger problem from the smaller one.

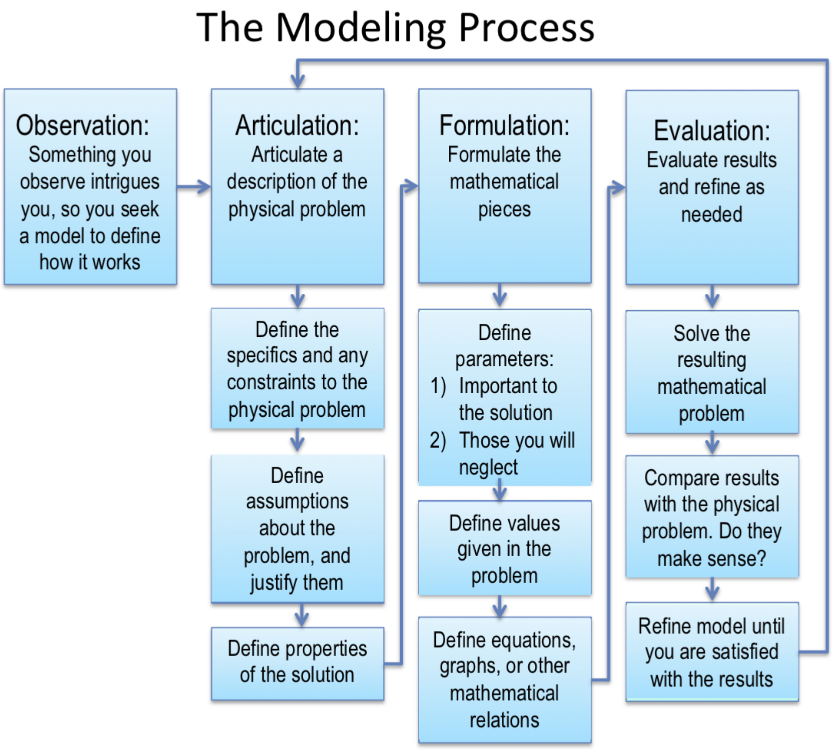
Some examples of mathematical models:

- Exponential and Logistic growth. You are familiar with the exponential and logistic growth models as they exist in function form (from algebra and calculus). However, these relationships are represented in differential equations also (as described above).
- Spring-mass system. Oscillatory motion is modeled by differential equations.
- Population models - Predator-Prey and Competitive Species, etc. Systems of differential equations model the interactions of populations.
- Epidemic models. Models of diseases are also represented by systems of differential equations.
- Heat transfer. When you make a cup of tea, coffee, or soup - you have the experience of the temperature change over time. However, we can use differential equations to model this temperature change.
- and MANY many more.

Mathematical models are formed through a modeling process. The modeling process outlines the steps and considerations to be used when creating a model. The modeling process is outlined in [**Fig. 1.1**].

Applied mathematics is a field in which we literally apply mathematics to real world problems. These problems are generally composed through the following:

1. Observed Phenomenon: something happens, we are intrigued, and want to examine the behavior or recreate it.
2. Mathematical model or data: we define a mathematical model for the phenomenon in an attempt to describe it, or we take data about what is happening in order to model it.
3. Solution: we solve the mathematical model by analytical or computational methods. (Differential equations is a subset of this step, Numerical Analysis covers some other approaches to these solutions.)



This image says "The Modeling Process" at the top and includes a flow chart shown in the figure contains boxes of text linked by arrows in the following order: Observation: Something you observe intrigues you, so you seek a model to define how it works -> Articulation: Articulate a description of the physical problem -> Define the specifics and any constraints to the physical problem -> Define assumptions about the problem and justify them -> Define properties of the solution -> Formulation: Formulate the mathematical pieces -> Define parameters: 1) Important to the solution 2) Those you will neglect -> Define values given in the problem -> Define equations, graphs, or other mathematical relations -> Evaluation: Evaluate results and refine as needed -> Solve the resulting mathematical problem -> Compare results with the physical problem. Do they make sense? -> If so: Sensitivity Analysis -> Refine model until you are satisfied. There is then an arrow from the last box that cycles back to the box that says Articulation: Articulate a description of the physical problem.

Figure 1.1: A flowchart to represent the Modeling Process.

4. Results: we relate these results to the observed phenomenon to make predictions, or modifications to our model and/or its implementation.

This course emphasizes the work done in solutions, but will connect those solutions to their models and the results in the world.

Differential Equations Models

In this course, you will need to be able to look at a given equation and describe what it is telling you about the relationships defined by the equation. So, given the earlier example with $P' = kP$, we discussed how that equation defines a rate of change in the population that is proportional to the current population by a rate k. So, how does the Logistic model compare? $P' = kP\left(1 - \frac{P}{N}\right)$?

In the Logistic model, there are two parameters k and N. k appears to be similar to the exponential model. N is defining a bound on the population. The term $1 - \frac{P}{N}$ will be positive for values of P that are less than N, and it will be negative if the value of P is larger than N. So, if you're familiar with the Logistic growth equation, N is the carrying capacity. kP defines a growth rate (or decay rate if k is negative), and the product with $1 - \frac{P}{N}$ ensures that the population approaches the carrying capacity as it grows. Notice that the rate of change in the population approaches zero as P approaches N. This means that the population does not change if P reaches N. We'll talk about this more when we determine equilibrium solutions.

Spring-Mass System In the Spring-mass system, we use the relationship of equal and opposite forces. The spring force (Hooke's Law) $F = -kx$ matched with Newton's second law $F = ma$ (the equal and opposite forces) as the foundation for the model. When the spring is moved off equilibrium, the mass experiences a force from the spring equal and opposite to the force used to hold it off equilibrium. Upon releasing the spring, there is an instantaneous velocity caused by the change from a static system to a dynamic one. The force from the spring is what results in an acceleration, and thus changes the position and velocity of the mass. The models for spring mass systems:

Undamped: the forces are directly matched $ma = -kx$, resulting in the equation $mx'' + kx = 0$

When looking at the undamped equation, we can manipulate the parameters to a single parameter (it's nonphysical to have a mass that is zero), we assume $m \neq 0$ and divide both sides by m.

$x'' + \frac{k}{m}x = 0$ or $x'' = -\frac{k}{m}x$

Reading this equation, we see that the acceleration of the mass is directly proportional to its position - but opposite in sign. The mass displacement from equilibrium is the cause of its acceleration. If the mass is pulled out from equilibrium, then it will accelerate back toward equilibrium at a rate determined by the ratio k/m (spring constant divided by the mass).

Damped: When the mass has contact with a surface, there is friction. The frictional force is directly proportional to the velocity of the mass. The spring force and the frictional force are working to move the mass back to equilibrium (the spring force reduces to zero when the mass is at equilibrium, and the frictional force works to slow the motion of the mass.) The external force used to move the mass off of equilibrium was then in opposition to both forces. Including a term with velocity and a parameter for the frictional force $-\gamma x'$ results in a relation $mx'' = -\gamma x' - kx$, or the commonly used form $mx'' + \gamma x' + kx = 0$

This equation can be manipulated in the same way, but is harder to read because it has three terms. Algebraically, we'll do the same operations as we did for the undamped equation.

$x'' + \frac{\gamma}{m}x' + \frac{k}{m}x = 0$ or $x'' = -\frac{\gamma}{m}x' - \frac{k}{m}x$

Here, we see the same proportional relationship with the displacement, but we also see a proportional relationship between the acceleration and the velocity. The mass will approach equilibrium faster because the acceleration will change with the velocity as well as the displacement. For example, once the mass is released it accelerates toward equilibrium - so the velocity is the same sign as the acceleration (at first, anyway, before it passes equilibrium.) If the displacement was positive, the acceleration became negative, which caused the velocity to become negative in the instants after. A negative velocity with a positive displacement will reduce the value of the acceleration faster than the undamped case. So, by adding the damping term, we see the acceleration change faster, and ultimately we see the mass return to equilibrium faster than an undamped case. It's all written in the equation, you can see the relationships defined by the equation, and draw conclusions about the behavior of the system from the equation. In mathematical modeling, we go the opposite direction - we use the physical relationships to define the terms in the mathematical model (similar to my discussion when we introduced the equations).

Newton's Law of Cooling Newton's Law of Cooling is an equation modeling the change in temperature of an object when placed in an environment that is another temperature. Most application problems are about a cup of tea or coffee, but some of them get more creative by using a pot of stew or a dead body.

Basically, you have the variable, T, which is the temperature of the object. Modeling the change in its temperature requires parameters for the ambient temperature, A, and the proportionality constant, k is formed heat transfer coefficient and surface area exposed. Ultimately, the model is formed by an assumption that the rate of change in the temperature of the object is directly proportional to the difference between the temperature of the object and its environment.

$$\frac{dT}{dt} = k(T - A)$$

The model is simpler than the others we've discussed, but there are some important takeaways from its simplicity. Notice, if $A > T$ then the temperature should increase (like an ice cube melting on your counter), but the difference $T - A$ would be positive. This form requires that k is negative for it to function appropriately. Alternatively, we can use $T' = k(A - T)$ if we want a positive proportionality constant.

The solution is an equation you worked with in Algebra: $T(t) = A + Ce^{kt}$ where C is defined by the initial temperature.

Systems of Differential Equations Models

Systems of differential equations involve multiple variables that are dependent on one variable (think back to vector functions in Calculus III - its a mind boggler.)

Best way to show this is examples. One of the common ones we'll discuss are population models: Predator-Prey and Competitive Species.

Predator-Prey

Predator-Prey involves two species in a closed system. The predator survives on the prey as their only food source and die off without prey if their population is too high. The prey are killed off by the predator, but have plenty of food to grow their population (without the predators, we assume their population would grow without bound.)

Putting these into an equation, we have to denote them mathematically. So, it's helpful to know what you're modeling. We'll use Blue Whales (Whales) and Krill, Whales are the predator that survives on the Krill, their prey. So, $W =$ Whales in our equation, and $K =$ Krill.

Without Krill, the Whales will die off. Using the knowledge we discussed about population models, it's natural to represent this through $W' = -aW$ where a is a parameter that defines the rate at which their population dies off.

With Krill, the Whales are able to survive at a different rate - a positive interaction term bKW (we use multiplication to denote an interaction of two species/variables). where b is another parameter to define the rate of growth with Krill. We combine their rate of decline without Krill with their rate of survival/increase with Krill.

$W' = -aW + bKW$

We also must model Krill. The Krill have enough food to survive and grow on their own $K' = cK$. However, they are dying at a rate that is defined through their interactions with the Whales - and their rate of death is not equivalent to the rate of growth for the Whales (Blue Whales eat up to four tons of Krill each day, for example.) If we define that interaction by $-dkW$, we then combine the terms to model the population of Krill.

$K' = cK - dKW$

The system of differential equations is then:

$$\frac{dW}{dt} = -aW + bKW$$

$$\frac{dK}{dt} = cK - dKW$$

Notice that the output variables are W and K, but the input variable is t for both equations. The equations are also dependent on each other for a solution.

Reading the equations, we see that the Whale population is declining, but the interactions with the Krill population increase their numbers. The Krill population is growing, but interactions with the whales decreases their population. We can evaluate them further by seeing that the Whale population would die off without the Krill, but the Krill population would thrive without the Whales. The natural assumption just from the form of these equations is: the Whales are killing off the Krill, likely for food.

Competing Species

Competing species are not eating each other. This is another closed system where the two interact with each other, but in this case they are competing for the same source of food (and that food is not part of the model).

We assume that there is some fixed amount of food, so these two species must compete for food in order to survive. Additionally, they have to compete within their own species for food.

First, both species have growth rates (due to reproduction) that are proportional to their populations (just like the exponential growth models.) Both species will have decay rates due to the competition (interaction) in their own species, and the competition with the other species.

To write this mathematically, let's use two species of whales that both eat Krill (remember, we aren't modeling the Krill this time). Blue Whales will be denoted by a B, and Humpback Whales will be denoted by an H. Both species rely on the Krill population, and they are in competition to consume this limited food source.

$B' = a_1B - c_1B^2 - d_1BH$

$H' = a_2H - c_2H^2 - d_2BH$

Reading these equations, we see that the equations for both species are much the same. Both populations are growing at a specified rate, but the interactions with each other and their own species are causing a decrease in that growth rate. Due to the context of the problem, we know that is due to a limited food source. If we manipulate the equations algebraically, we can see that these are both similar to the logistic growth model we discussed earlier.

$B' = a_1B(1 - \gamma_1 B - \delta_1 H)$

$H' = a_2H(1 - \gamma_2 H - \delta_2 B)$

Where each γ is formed by c/a and each δ is formed by d/a in each equation. Both are facing logistic growth while in competition with each other for the overall population that can be supported by the Krill available.

Epidemics

Epidemic models are used to model the spread of a disease among a population. We divide the population into groups, Susceptible, Infected, and Recovered (though, I'm old school and prefer the previously used Removed). Susceptible members are able to get the disease, Infected members are currently infected with the disease, and Recovered members are not able to be infected (typically due to death or immunity.) This is referred to as the SIR model, the basic model uses a closed system of a fixed population (no births or deaths change the total population.)

Let's use the SIR model as a quick check for your ability to read a differential equation (in this case, a system of differential equations.)

$S' = -bSI$
$I' = bSI - cI$
$R' = cI$

What do these equations tell you about the populations of Susceptible, Infected, and Recovered (Removed)? What do you expect will happen?

In the basic model, because the population is fixed - Susceptibles are infected at a fixed rate determined by their interaction with Infected members. The Infected population grows at the same rate, but decreases as Infected members recover from the disease. When the Infected recover, they are moved to the Recovered group. Eventually, in this very basic model, all members will get infected (unless the infected population becomes zero), and then recover. So, eventually everyone is recovered (through death or immunity.)

The nice thing about this basic model is that it is merely a foundation. We can modify it to meet any desired features we need. Need to include a birth rate? reinfection rate? effects of a vaccine? We can add to it.

Birth rate: add terms, like rS for a birth rate of our Susceptible population.

Reinfection rate: Some (or all) members of the Infected population return to Susceptible instead of Recovered. When there is no immunity or death, it is common to reduce this model to an SIS model - the populations just alternate between Susceptible and Infected at defined rates.

Vaccination: Introduce another group, Vaccinated. Then, the rates of infection will be different for this group based on the vaccine effectiveness.

Mathematical modeling opens up a door to problem solving that you may not have known existed. Hopefully, this has given you some insight into how these models work and what they mean. When solving new problems, mathematical modeling is necessary. Next we will begin to solve the mathematical problems that result from these models.

1.1.3 Verifying Solutions of Differential Equations

We will not be solving equations in this section, but we can use the tools from Calculus to verify solutions. Given a "solution", use your derivative rules and substitute the values into the given equation - if the equation is true (the left side is equal to the right side), then the solution given is a solution to the differential equation. If the equation doesn't work, for example: $\sin(t) = 0$, $1 = 0$, or any case where the two sides are not the same - then that solution does not satisfy the equation and it is **not** a solution to that equation.

■ **Example 1.1** Verify that $y = t^2 + 2$ is a solution to $y'' - y = 0$.

First, take derivatives of y. The given equation is second order, and y is a function of t, so we need to determine the second derivative of y with respect to t.

$y' = 2t$
$y'' = 2$

Substituting the values of y, y', and y'' where they are in the equation leads to:

$y'' - y = 2 - (t^2 + 2) = -t^2$

$-t^2$ is not equal to zero, so this solution is **not** a solution to the given ODE. When we attempted to verify the solution, it didn't work. ■

■ **Example 1.2** Verify that $y = 2\sin(t)$ is a solution to $y'' + y = 0$.

We will again define the derivatives of y in the equation.

$y' = 2\cos(t)$

$y'' = -2\sin(t)$

$y'' + y = -2\sin(t) + 2\sin(t) = 0$ The combination of terms simplifies to zero, in this case. $0 = 0$, so this solution does satisfy the equation, and is a solution to the ODE. ■

Initial Value Problems

Initial value problems (IVPs) involve an initial condition on the function. Technically, an initial value problem just needs a value of the function - it doesn't have to be when $t = 0$ (but that is usually the most sensible choice). We generalize the definition as $t = t_0$ to represent the initial time, or the start time. In applications, we use the initial condition from a measurement and then use the ODE model to determine the expected behavior after the initial condition. Mathematically, we use it to define the function representing that behavior.

In the last example, we could have used any coefficient $y = A\sin(t)$, not just 2. So, $y = A\sin(t)$ is what we refer to as a general solution to the ODE. It's a general solution because it defines the form of solution, but there are still undefined parameters in the solution, A. The general solution defines what we refer to as a family of solutions (this family has infinite members, so just imagine that family gathering.) In the qualitative analysis section, we will work to depict the picture formed by these general solutions. A particular solution is defined to be the one solution that satisfies the given initial value(s). Each particular solution forms a solution curve (integral curve), which we will plot when we are studying qualitative analysis. In our example, an initial value is used to define the value of A. Initial values are commonly when $t = 0$ because we naturally use the starting point as $t = 0$. Initial values can also be given on derivative values, not just the function.

In our example, there's no way to define A when $t = 0$ using the solution, the Sine function is zero when $t = 0$. So, we can define A with an initial condition on y' when $t = 0$, or through an initial condition that is not at $t = 0$. For example, if $y'(0) = 5$, we could then use $y' = A\cos(t)$, substitute $t = 0$ into the derivative and see that $y'(0) = A$. Given the condition $y'(0) = 5$, A must equal 5.

Alternatively, if we were given a condition $y(\pi/2) = 3$, we would use $y = A\sin(t)$. We evaluate $y(\pi/2) = A\sin(\pi/2) = A$, and combined with the condition find that $A = 3$ in that case.

■ **Example 1.3** Verify that $y(t) = 2e^{5t}$ is a solution to the initial value problem (IVP): $y'' - 5y' = 0$, $y(0) = 2$.

First, determine the derivatives.

$y'(t) = 10e^{5t}$

$y''(t) = 50e^{5t}$

Substitute them into the given equation to verify.

$y'' - 5y' = 50e^{5t} - 5 * 10e^{5t} = 50e^{5t} - 50e^{5t} = 0\checkmark$

The solution satisfies the equation. Now, check that it satisfies the initial condition.

$y(0) = 2e^{5*0} = 2e^0 = 2 * 1 = 2\ \checkmark$

The solution also satisfies the initial condition, so $y = 2e^{5t}$ is a solution to this initial value problem. ■

These problems can become more complicated through higher derivatives and/or additional parameters. Here's a common example with a second parameter.

■ **Example 1.4** Verify that $y(t) = 2\cos(4t) + 3\sin(4t)$ is a solution to the IVP $y'' + 16y = 0$ with initial conditions $y(0) = 2$ and $y'(0) = 12$.

We again start by determining the derivatives and substituting them into the equation.

$y'(t) = -8\sin(4t) + 12\cos(4t)$
$y''(t) = -32\cos(4t) - 48\sin(4t)$
$y'' + 16y = -32\cos(4t) - 48\sin(4t) + 16\,(2\cos(4t) + 3\sin(4t))$
$= -32\cos(4t) - 48\sin(4t) + 32\cos(4t) + 48\sin(4t) = 0\checkmark$

Then, we check the initial conditions.
$y(0) = 2\cos(0) + 3\sin(0) = 2 + 0 = 2\ \checkmark$
$y'(0) = -8\sin(0) + 12\cos(0) = 0 + 12 = 12\ \checkmark$

The given solution satisfies both the equation and the initial conditions, so it is a solution to the given IVP.

*This is also an example of an undamped oscillator. In this problem it's initial displacement from equilibrium is 2 spatial units (could be cm, mm, inches, feet - all depends on context), with an initial velocity of 12 spatial units/time unit (the time units could be in seconds, minutes, hours, etc. Again, all depends on context.) A common unit choice is *cm* and *s* (seconds). ∎

Boundary Value Problems

Boundary Value Problems (BVPs) use boundary values (as you might have guessed.) BVPs have two conditions on the function, or its derivatives. We will not do many problems involving boundary values, they are covered more in Partial Differential Equations. Classic problems, like a vibrating string with two fixed ends, have boundary values (the location of the two ends). Boundary values are typically physical constraints on the problem in space, rather than time. There are problems that bend the rules for mathematical practice, but typically in applications an IVP is defined with respect to time, and a BVP is defined with respect to space.

1.2 Separable Equations

Your first solution technique! Separable equations rely on a relationship implied through the notation $\frac{dy}{dt}$. If you think back to your calculus course, when you first learned derivatives, you evaluated $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. When I teach calculus, I introduce this as a small $\frac{\text{change in } y}{\text{change in } x}$. You may also remember that the dy and dx notation were used as differentials in Calculus I. Now, our equations are using t in place of x, but variable names are just placeholders - we can use anything there. We could look at $Mountain(Bob)$ or $Danger(Explosives)$, and the relationships defined mathematically would remain the same. All of that to say that we can treat that notation like division... what does that mean? Good question.

When I have a differential equation like $\frac{dy}{dt} = ty$, or any general $\frac{dy}{dt} = p(t)q(y)$, I can rewrite it as $\frac{dy}{y} = t\,dt$, or $\frac{dy}{q(y)} = p(t)dt$. This is a separable equation, because I can manipulate all terms of y to one side, and all terms of t to the other side. Thereby separating the y's and the t's.

Then, I can use all those fancy integration techniques from Calculus II to integrate both sides with respect to their variables, and simplify to solve for the function $y(t)$.

This example isn't super complicated, but if you remember how difficult some integration problems were in Calculus II, then you know it can get hard to solve these.

Let's first emphasize what is separable, and what is not separable.

- $y' = ty$ is separable.
- $y' = t + y$ is **not** separable.
- $y' = 2t\sin(t)$ is separable.
- $y' = 2y(1-y)$ is separable.
- $y' = 4y(1-t)$ is separable.
- $y' = 1$ is separable.

- $y'' = y$ is **not** separable. It isn't a first-order equation.
- $y' = 2te^t y$ is separable.
- $y' = 1 + ty$ is **not** separable.

Show yourself that these statements are true by trying to separate your y's and t's in each equation. Replace y' with $\frac{dy}{dt}$ and manipulate the equations to separate the variables.

Hopefully, you are finding some comfort with this concept after working through that exercise. The rest, is calculus/integration.

Let's solve some problems.

■ **Example 1.5** Solve $y' = 2y$ using separation of variables.

First, rewrite it as $\frac{dy}{dt} = 2y$.

Manipulate the equation so y terms are on the left, and t terms are on the right.

$\frac{dy}{2y} = dt$

Then integrate both sides of the equation. Recall: you need a $+C$, but since you are setting two integrals equal to each other, only one of them needs a $+C$ this time. We use the t side for consistency and ease of further algebraic manipulation.

$\int \frac{dy}{2y} = \int dt$

$\frac{1}{2} \ln|y| = t + C$

This is the solution, but it's still in an implicit form (an explicit form has $y(t) =$ notation.) We do a bit of algebra to get it to explicit form.

$\ln|y| = 2t + 2C$

$y(t) = e^{2t+2C}$

We use our knowledge of exponential functions to simplify this further, and define $k = e^{2C}$ to simplify the notation.

$y(t) = ke^{2t}$ (this is our general solution!)

To determine a particular solution, we need an initial condition. If we were given one, $y(0) = 37$, then we would substitute zero for t and determine k.

$y(0) = ke^0 = k$, so $k = 37$.

The particular solution is then $37e^{2t}$. ■

■ **Example 1.6** Solve the initial value problem using a separation of variables. $\frac{dy}{dt} = 2t\sin(t)y$, $y(0) = 1$.

First, separate the variables.

$\frac{dy}{y} = 2t\sin(t)dt$

This example will require integration by parts. Integrate both sides of the equation.

$\int \frac{dy}{y} = \int 2t\sin(t)dt$

$\ln|y| = -2t\cos(t) + 2\sin(t) + C$ (implicit solution. We can use this form to solve for C)

$y(t) = e^{-2t\cos(t)+2\sin(t)+C} = ke^{-2t\cos(t)+2\sin(t)}$ $(k = e^C)$

Then, applying our initial condition $y(0) = 1$.

If we use the implicit form, we have: $\ln|1| = -2 * 0\cos(0) + 2\sin(0) + C$.

Or, $0 = 0 + 0 + C$, so $C = 0$.

The *implicit solution* is $\ln|y| = -2t\cos(t) + 2\sin(t)$ because $C = 0$.

If we use the explicit form, we have: $y(0) = ke^{-2*0\cos(0)+2\sin(0)} = ke^0 = k$, so $k = 1$.

The *explicit solution* is $y(t) = e^{-2t\cos(t)+2\sin(t)}$ because $k = 1$.

These two solutions are consistent because the relationship between y and t is the same. We can verify that the constants are consistent also $C = 0$ and $k = e^C$, $k = e^0 = 1$. This is why it is important to track how each constant is named and formed, if we'd tried to use zero for k we would have had a problem. ■

■ **Example 1.7** Determine if the following ODE is separable. If it is, determine the solution to the IVP. $y' = ty - t$, $y(0) = 4$.

Is it separable? There's a t in both terms, let's see if it factors to a product of two functions - one in t and one in y.

[fix the typo, it's $y-1$, not $1-y$.] $y' = t(y-1)$ yes, it's separable.

Commence with a separation of variables.

$\frac{dy}{dt} = t(y-1)$

$\frac{dy}{y-1} = tdt$

Integrate both sides of the equation.

$\int \frac{dy}{y-1} = \int tdt$

U-substitution: $u = y-1$, $du = dy$

$\int \frac{du}{u} = \int tdt$

$\ln|u| = \frac{t^2}{2} + C$

$\ln|y-1| = \frac{t^2}{2} + C$ (implicit general solution)

$\ln|y-1| = \frac{t^2}{2} + C$

$y - 1 = e^{t^2/2+C}$

$y(t) = 1 + e^{t^2/2+C}$

$y(t) = 1 + ke^{t^2/2}$, $k = e^C$ (explicit general solution)

Apply the initial value, $y(0) = 4$

$y(0) = 1 + ke^0 = 1 + k$, so $1 + k = 4$, or $k = 3$.

$y(t) = 1 + 3e^{t^2/2}$ (explicit particular solution)

Implicit form: $\ln|4-1| = 0 + C$, or $C = \ln(3)$.

$\ln|y-1| = \frac{t^2}{2} + \ln(3)$

Check for consistency: $k = e^C = e^{\ln(3)} = 3$. ✓Yes, these are consistent.

Why might these be inconsistent? The absolute value in the antiderivative to ensure that the logarithm exists. If the initial value is causes the term inside the logarithm to be negative, without the absolute value, then the two forms may be inconsistent. If this is the case, solve for the explicit solution before applying your initial value. However, for most problems this will not be an issue.

If we simplify this solution to an explicit form, we get

$(y-1)/3 = e^{t^2/2}$

$y = 1 + 3e^{t^2/2}$ consistent.

When solving initial value problems, it's safest to convert them to explicit form before applying the initial value to ensure consistency. However, it isn't always necessary. ■

■ **Example 1.8** Solve the logistic population model for an island of rats with a growth rate of 20% per year, an initial population of 300, and a carrying capacity of 5000.

First, lets construct the equation using the principles from section 1.

The growth rate is 20%, so the rats will increase at a rate of $0.2P$ (where P is representing the population of rats).

The carrying capacity is 5000, so we have to ensure that the population rate of change decreases as it approaches 5000. The term $\left(1-\dfrac{P}{5000}\right)$ ensures that the rate of change approaches zero as the population reaches the carrying capacity.

So, $\dfrac{dP}{dt}=0.2P\left(1-\dfrac{P}{5000}\right)$.

The initial population is our initial value, $P(0)=300$.

So, we solve for $P(t)$. This is a separable equation.

$\dfrac{dP}{0.2P(1-P/5000)}=dt$

$\int \dfrac{dP}{0.2P(1-P/5000)}=\int dt$

This will require partial fraction decomposition.

$\dfrac{1}{0.2P(1-P/5000)}=\dfrac{A}{0.2P}+\dfrac{B}{1-P/5000}$

Multiplying both sides by the left denominator yields

$1=A(1-P/5000)+B(0.2P)$

$1=A-AP/5000+0.2BP$

Clearly, $A=1$ (there's no P terms on the left side, so we set those terms equal to zero).

So, $0=-AP/5000+0.2BP$, but $A=1$ and P is in both terms, so we can factor it out.

$0=-1/5000+0.2B$

$1/5000=0.2B$

$1/1000=B$

So, $\dfrac{1}{0.2P(1-P/5000)}=\dfrac{1}{0.2P}+\dfrac{1}{1000(1-P/5000)}=\dfrac{5}{P}+\dfrac{1}{1000-0.2P}$

Each of these terms we can integrate separately.

$\int \dfrac{dP}{0.2P(1-P/5000)}=\int\left(\dfrac{5}{P}+\dfrac{1}{1000-0.2P}\right)dP=5\ln|P|-5\ln|1000-0.2P|=5\ln\left|\dfrac{P}{1000-0.2P}\right|$

Combining this with the right side: $5\ln\left|\dfrac{P}{1000-0.2P}\right|=\int dt=t+C$ (implicit solution)

Algebra time! $\ln\left|\dfrac{P}{1000-0.2P}\right|=\dfrac{t+C}{5}$

$\dfrac{P}{1000-0.2P}=e^{(t+c)/5}$

$P=e^{(t+C)/5}(1000-0.2P)=1000e^{(t+C)/5}-0.2Pe^{(t+C)/5}$

$P+0.2Pe^{(t+c)/5}=1000e^{(t+C)/5}$

$P\left(1+0.2e^{(t+C)/5}\right)=1000e^{(t+C)/5}$

$P=\dfrac{1000e^{(t+C)/5}}{1+0.2e^{(t+C)/5}}=\dfrac{5000}{5e^{-(t+C)/5}+1}$

This is the form of our Logistic equation from algebra, $P(t)=\dfrac{5000}{1+5e^{(t+C)/5}}$

Applying the initial condition $P(0)=300$

$P(0)=\dfrac{5000}{1+5e^{C/5}}=300$

Solving for C, $\dfrac{50}{3}=1+5e^{C/5}$

$\dfrac{47}{3}=5e^{C/5}$

$\dfrac{47}{15}=e^{C/5}$

$\ln\left(\dfrac{47}{15}\right)=\dfrac{C}{5}$

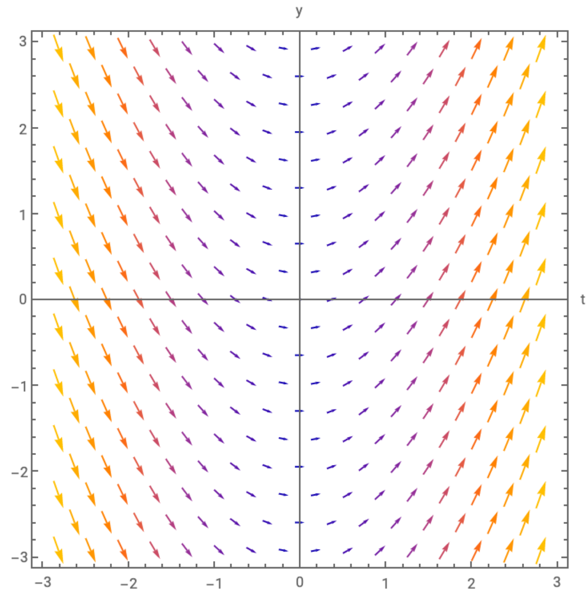


Figure 1.2: This is a direction field plot generated by Mathematica for $f(t,y) = t$. When sketching solution curves, you will draw them by hand into the direction field plots.

$$5\ln\left(\frac{47}{15}\right) = C$$

Substituting C back into our solution, $P(t) = \dfrac{5000}{1+5e^{(t+5\ln(47/15))/5}}$ ■

Now, practice with the examples listed earlier in this section.

1.3 Geometry and Quantitative Analysis

In this section we will focus on the visual representation of our differential equations and their solution curves.

1.3.1 Direction Fields

Direction fields are vector fields that sketch the slope at each point in space. We construct direction fields for first order differential equations using the definition of our derivative to define the vector field.

■ **Example 1.9** Sketch the direction field for $\dfrac{dy}{dt} = t$

For each point (t,y) in the plane, we use the function $f(t,y) = t$ to define the slope at that point. Then, we sketch a vector from that point in the direction of the slope.

So, when $t = 0$, $f(0,y) = 0$, so all points along that line have the same slope of zero. We sketch in a dot for a zero vector. When $t = 1$, $f(1,y) = 1$, so all points along that line have the same slope of 1. We sketch in a vector to represent the slope of 1 (same rise as run), which is parallel to the line $y = t$ $(dy/dt = 1.)$ This continues, until we have a rough picture of the slopes at every point. [insert sketch] Once you've sketched parts by hand, a computer is useful for a more complete visual. ■

■ **Example 1.10** Sketch the direction field for $\dfrac{dy}{dt} = t + y$

For each point (t,y) in the plane, we use the function $f(t,y) = t+y$ to define the slope at each point. Then, we sketch a vector from that point in the direction of the slope.

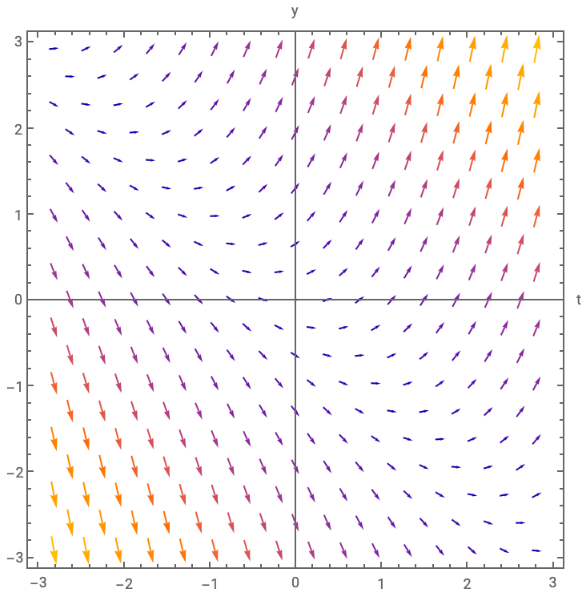


Figure 1.3: This is a direction field plot generated by Mathematica for $f(t,y) = t + y$. When sketching solution curves, you will draw them by hand into the direction field plots.

In this example, it isn't solely determined by the value of t, there is also dependence on y. When sketching these by hand, pick a range of points, and no less than 5. If you get values where the slope is zero, add another point to your table. You pick the points, and you can pick any values of t and y you like - but it is wise to select points near the origin, and/or near the solution you seek.

t	y	dy/dt	vector (to be normalized)
0	0	0	a vector in the direction of $\langle 1,0 \rangle$ from $(0,0)$
1	0	1	a vector in the direction of $\langle 1,1 \rangle$ from $(1,0)$
2	1	3	a vector in the direction of $\langle 1,3 \rangle$ from $(2,1)$
-1	1	0	a vector in the direction of $\langle 1,0 \rangle$ from $(-1,1)$
-1	3	2	a vector in the direction of $\langle 1,2 \rangle$ from $(-1,3)$
1	-1	0	a vector in the direction of $\langle 1,0 \rangle$ from $(1,-1)$
2	-1	1	a vector in the direction of $\langle 1,1 \rangle$ from $(2,-1)$
-1	-1	-2	a vector in the direction of $\langle 1,-2 \rangle$ from $(-1,-1)$

[insert sketch]

Once you've sketched parts by hand, a computer is useful for a more complete visual. ■

1.3.2 Solution Curves

Solution curves (also called integral curves) give you a picture of a particular solution. If we think of direction fields providing a visual of the behavior(s) of the general solution, the solution curves are the particular solution. Once we have a direction field, we can use it to sketch in solution curves. We use the point given by the initial value and then follow the behavior of the direction field to sketch in the associated solution curve.

■ **Example 1.11** Using the first direction field, for $dy/dt = t$, sketch in solutions for the following initial values:

a) $y(0) = 2$
b) $y(0) = -1$
c) $y(0) = 0$

Once you've sketched parts by hand, a computer is useful for a more complete visual. To sketch in solution curves for each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve. [add sketch of solution curves on direction field] ■

■ **Example 1.12** Using the second direction field, for $dy/dt = t + y$, sketch in solutions for the following initial values:
a) $y(0) = -2$
b) $y(0) = 1$
c) $y(0) = 0$

Once you've sketched parts by hand, a computer is useful for a more complete visual. To sketch in solution curves for each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve. [add sketch of solution curves on direction field] ■

1.3.3 Autonomous DIfferential Equations

As was earlier described, an autonomous differential equation is a first-order equation where the derivative has no dependence on t. Instead of the general function $dy/dt = f(t,y)$, an autonomous differential equation is of the form $dy/dt = f(y)$.

In these equations, the direction field looks the same for all t, so the slope is determined for several y values, and then repeated in horizontal strips in the direction field (which makes it easier to sketch. Note, in our first direction field, it was only dependent on t and the slopes were repeated in vertical strips.)

For autonomous equations, one of the remarkable features is that a zero slope causes a flat line as $t \to \infty$. These are equilibrium solutions, because if a curve approaches one of these lines, it will stay with it.

Equilibrium Solutions

Equilibrium solutions are values where the derivative of a function is equal to zero. If we recall from Calculus I, the points where a derivative is zero were called critical points (or critical values). In Calculus, we used these points to classify the critical points (maximum, minimum, inflection point). In differential equations, we will discuss these points in terms of the stability of our solutions. One distinct difference here is that we would look to the left and right of a critical point in Calculus (using t values), but for an autonomous equation we are looking above and below (using y values). In Calculus, the function's derivative would always be in terms of the independent variable, t, which made the analysis significantly easier. Equilibria in differential equations relate to the long-term behavior of solutions, not just a single point on a curve. The equilibrium solutions are not defined specific to one function, but rather a general behavior of all possible functions that satisfy the differential equation. So, there is no direct translation from what we did in Calculus I to what we are doing now - but there are some similarities.

We are currently studying equilibrium solutions for autonomous differential equations. The Logistic equation is an autonomous differential equation (it's a first order differential equation that only depends on the function, P, and has no dependence on its input variable), so it is a good example to work through.

In the Logistic equation there are two equilibrium solutions.

$P' = kP\left(1 - \frac{P}{N}\right) = 0$ when $P = 0$ and when $P = N$.

We can determine the stability of these solutions by evaluating the sign of our derivative near each equilibrium point (solution).

If $P < 0$, then $P' < 0$. If $0 < P < N$, then $P' > 0$. On a phase line, we draw the points on a vertical line then add arrows to show if the derivative is increasing (up) or decreasing (down)

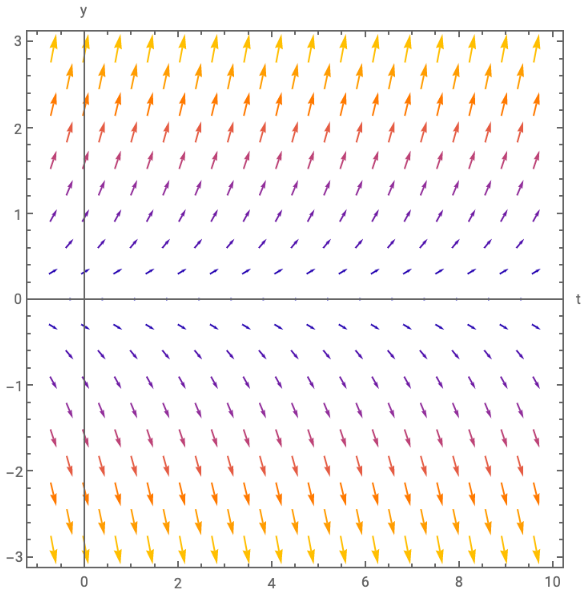


Figure 1.4: This is a direction field plot generated by Mathematica for $f(t,y) = y$. When sketching solution curves, you will draw them by hand into the direction field plots. For each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve.
[add solution curve sketches]

near each point. For this problem, the arrows both point away from $P = 0$ which means that the equilibrium solution is **unstable**, and we call that point a **source**. Any value of P that is not zero will move further away from zero.

If $N < P$, then $P' < 0$. So, combining that with the earlier check: if $0 < P < N$, then $P' > 0$. We find that the point at $P = N$ has both arrows pointing toward it. Any value of P close to N will approach the value of N, so that equilibrium solution is **asymptotically stable**, and we call that point a **sink**.

The third possibility occurs when the arrows point in the same direction above and below an equilibrium point (both arrows point up, or both arrows point down). In this case, the point is stable from one direction, but unstable in the other direction, which means that the equilibrium solution **semistable**, and we call that point a **node**.

[add example phase line diagrams]

A lot of the content in this section focuses on qualitative analysis of the solutions. Qualitative analysis uses descriptive language to describe the solutions like stable, unstable, or semistable. The quantitative analysis side involves the quantities associated with the solution. So, if there's an equilibrium point at $y = 0$, and it is stable (quality), then we can say that solutions near $y = 0$ would trend toward it as $t \to \infty$ (quantity). Connecting the quality to the quantitative meaning is important context for the work we are doing. When we cannot solve the solution using our current techniques, these qualitative analyses allow us to say *something* about the solutions.

Let's evaluate some examples of autonomous differential equations: their direction fields, how the equilibrium solutions are represented, and how the solution curves are affected.

■ **Example 1.13** Sketch the direction field for $\dfrac{dy}{dt} = y$, solution curves for $y(0) = -2$, $y(0) = 0$, and $y(0) = 1$. Classify any equilibrium points, and describe how the solution curves were affected. The only equilibrium point occurs when $y = 0$. Since the phase line points away from $y = 0$ (negative y values result in negative slope, and positive y values result in positive slope), this equilibrium point

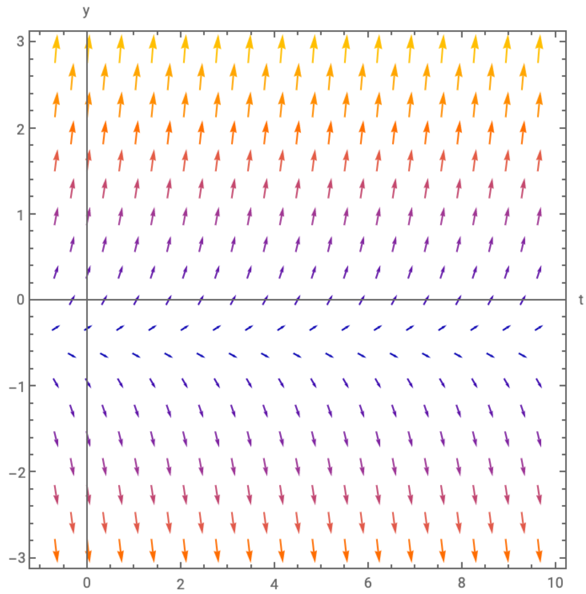


Figure 1.5: This is a direction field plot generated by Mathematica for $f(t,y) = 2y+1$. When sketching solution curves, you will draw them by hand into the direction field plots. For each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve.
[add solution curve sketches]

is an unstable source. [add phase line]

The solution curves show that $y(0) = -2$ will cause the solution to approach negative infinity as $t \to \infty$, $y(0) = 0$ remains at zero as $t \to \infty$, and $y(0) = 1$ will grow to positive infinity as $t \to \infty$. Three different behaviors for each initial value, but they are indicative of the behavior of the family. If the initial value is below $y = 0$, then the solution will trend toward negative infinity. If the initial value is $y = 0$, then the solution stays on $y = 0$ (the only case where the long term behavior is non-infinite.) If the initial value is above $y = 0$, then the solution will trend toward positive infinity.

All to show that these equilibrium points are important when defining the behavior of your solution (even if you cannot solve it exactly). In this case, we can solve the equation for $y(t) = ke^t$. The relations we defined become clear when the initial value defines the value of k in this solution. When k is negative, then the exponential e^t grows toward infinity rapidly, but when multiplied by a negative value the overall solution trends toward negative infinity. Similarly, if $k = 0$ then the solution is identically zero. If k is greater than zero, the solution grows to positive infinity. ■

■ **Example 1.14** Sketch the direction field for $\dfrac{dy}{dt} = 2y+1$, solution curves for $y(0) = -2$, $y(0) = -1/2$, and $y(0) = 0$. Classify any equilibrium points, and describe how the solution curves were affected.

The only equilibrium point here is when $2y+1 = 0$, or $y = -1/2$. If we sketch the phase line, points below $y = -1/2$ are negative (down), and points above are positive (up). As in the last example, the equilibrium point at $y = -1/2$ is an unstable source. [add phase line]

The behaviors are the same (even though the direction field is not.) Initial values above $y = -1/2$ will grow to infinity, at $y = -1/2$ will not change (the solution is fixed at $-1/2$), and below will grow to negative infinity. The equilibrium point is where the solution changes. ■

■ **Example 1.15** Sketch the direction field for $\dfrac{dy}{dt} = y(y+2)$, solution curves for $y(0) = -3$, $y(0) = -2$, $y(0) = -1$, $y(0) = 0$, and $y(0) = 1$. Classify any equilibrium points, and describe how

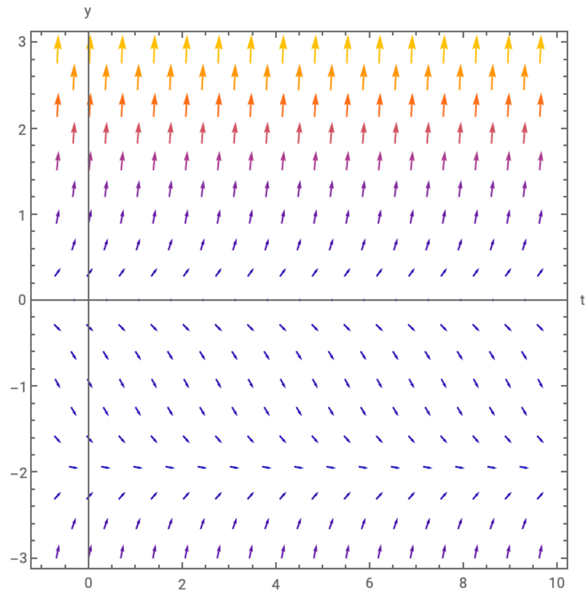


Figure 1.6: This is a direction field plot generated by Mathematica for $f(t,y) = y(y+2)$. When sketching solution curves, you will draw them by hand into the direction field plots. For each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve.
[add solution curve sketches]

the solution curves were affected.

Now we have two equilibrium points, one at $y = 0$ and one at $y = -2$. The phase line evaluates the behavior above and below each equilibrium point. If we evaluate our derivative at $y = -3$, $y = -1$, and $y = 1$ we'll be able to sketch the phase line clearly. [add phase line]

$f(t,-3) = -3(-3+2) = 3 > 0$ the arrow below $y = -2$ will point upward.
$f(t,-1) = -1(-1+2) = -1 < 0$, the arrow between $y = -2$ and $y = 0$ will point downward.
$f(t,1) = 1(1+2) = 3 > 0$, the arrow above $y = 0$ will point upward.

The arrows point toward $y = -2$, so it is an asymptotically stable sink. The arrows point away from $y = 0$, so it is an unstable source.

For the initial values given:
$y(0) = -3$ will approach $y = -2$ as $t \to \infty$.
$y(0) = -2$ will remain at $y = -2$ as $t \to \infty$.
$y(0) = -1$ will approach $y = -2$ as $t \to \infty$.
$y(0) = 0$ will remain at $y = 0$ as $t \to \infty$.
$y(0) = 1$ will grow, approaching positive infinity as $t \to \infty$.

Notice that an asymptotically stable equilibrium value is approached from both sides as $t \to \infty$. An unstable equilibrium value will only be reached if the function value matches it exactly, otherwise it will trend away from the equilibrium value. ■

■ **Example 1.16** Sketch the direction field for $\dfrac{dy}{dt} = y^2(4-y^2)$ and solution curves for $y(0) = -3$, $y(0) = -2$, $y(0) = -1$, $y(0) = 0$, $y(0) = 1$, $y(0) = 2$, and $y(0) = 3$. Classify any equilibrium points, and describe how the solution curves were affected.

This problem has three equilibrium points. $y = 0$, -2, and 2. We can evaluate their stability by looking at the rate of change near each value (above and below).

$f(t,-3) = (-3)^2(4-(-3)^2) = 9*(-5) = -45 < 0$ (downward arrow when $y = -3$.)
$f(t,-1) = (-1)^2(4-(-1)^2) = 1*3 = 3 > 0$ (upward arrow when $y = -1$.)

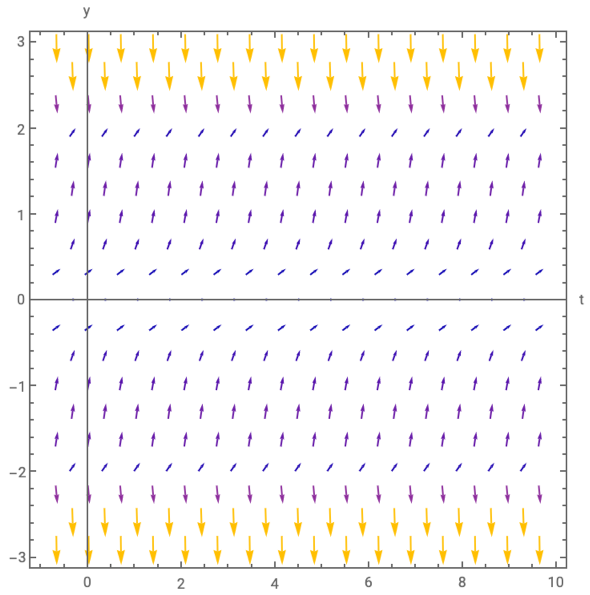


Figure 1.7: This is a direction field plot generated by Mathematica for $f(t,y) = y^2(4-y^2)$. When sketching solution curves, you will draw them by hand into the direction field plots. For each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve.
[add solution curve sketches]

$f(t,1) = 1^2(4-1^2) = 1*3 = 3 > 0$ (upward arrow when $y = 1$.)
$f(t,3) = 3^2(4-3^2) = 9*(-5) = -45 < 0$ (downward arrow when $y = 3$.)

The arrows above and below $y = -2$ point away from the equilibrium point, so it is an unstable source. The arrows above and below $y = 0$ point upward. This means that $y = 0$ is a semistable node, it is asymptotically stable on one side and unstable on the other side. The arrows above and below $y = 2$ point toward the equilibrium point, so it is an asymptotically stable sink. [add phase line]

The unstable solution only occurs when the function value is equal to $y = -2$, below that point the solution approaches $-\infty$ and above it the solution will approach zero. The semistable solution at $y = 0$ occurs only if solutions come from values between -2 and 0 (including 0, but not including -2.) The stable solution at $y = 2$ is the approached for all values greater than zero. ■

■ **Example 1.17** Sketch the direction field for $\frac{dy}{dt} = y(y+1)(y-2)$, solution curves for $y(0) = -2$, $y(0) = -1$, $y(0) = -1/2$, $y(0) = 0$, $y(0) = 1$, $y(0) = 2$, and $y(0) = 3$. Classify any equilibrium points, and describe how the solution curves were affected.

This problem has three equilibrium points, $y = 0$, -1, and 2. We can evaluate points near these to determine their stability.

$f(t,-2) = -2(-2+1)(-2-2) = -2*(-1)*(-4) = -8 < 0$ (downward arrow at $y = -2$.)
$f(t,-1/2) = -1/2(-1/2+1)(-1/2-2) = -1/2*(1/2)(-3/2) = 3/8 > 0$ (upward arrow at $y = -1/2$.)
$f(t,1) = 1(1+1)(1-2) = 1*2*(-1) = -2 < 0$ (downward arrow at $y = 1$.)
$f(t,3) = 3(3+1)(3-2) = 3*4*1 = 12 > 0$ (upward arrow at $y = 3$.)

The arrows above and below $y = -1$ point away from the point, so it is an unstable source. The arrows above and below $y = 0$ point toward the point, so it is an asymptotically stable sink. The arrows above and below $y = 2$ point away from the point, so it is an unstable source. [add phase

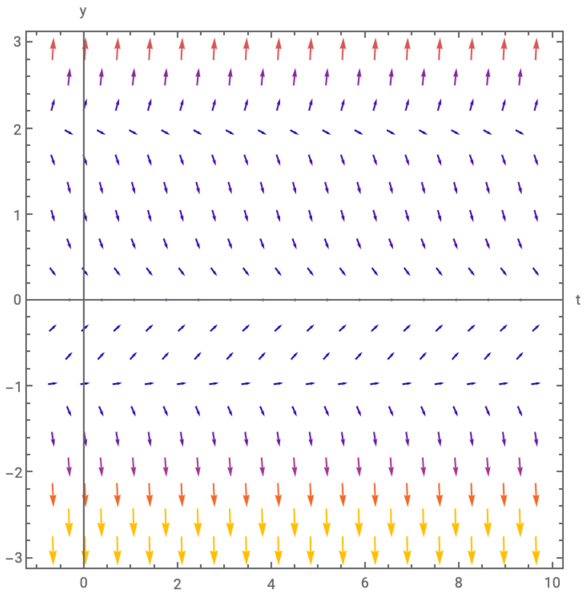


Figure 1.8: This is a direction field plot generated by Mathematica for $f(t,y) = y(y+1)(y-2)$. When sketching solution curves, you will draw them by hand into the direction field plots. For each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve.
[add solution curve sketches]

line]

As in the previous examples, values below $y = -1$ will grow toward negative infinity as $t \to \infty$. Values between $y = -1$ and $y = 2$ will asymptotically approach $y = 0$ as $t \to \infty$, except if they start at $y = 0$ (in which case they remain at zero.) Values above $y = 2$ will grow toward positive infinity as $t \to \infty$. ■

■ **Example 1.18** Sketch the direction field for $\frac{dy}{dt} = e^y - 1$, solution curves for $y(0) = -2$, $y(0) = 0$, and $y(0) = 1$. Classify any equilibrium points, and describe how the solution curves were affected.

This example is to serve as an acknowledgement that, while polynomials are easiest to evaluate, the $f(t,y)$ function can be *anything*. This equation has one equilibrium point when $y = 0$.

We can still evaluate the function above and below the equilibrium point to classify it.

$f(t,-1) = e^{-1} - 1 < 0$ (downward arrow below $y = 0$.)

$f(t,1) = e^1 - 1 > 0$ (upward arrow above $y = 0$.)

The equilibrium point at $y = 0$ is an unstable source. The solution curves will approach negative infinity for initial values below $y = 0$, and positive infinity for initial values above $y = 0$. The only defined long-term behavior occurs if the initial value is equal to $y = 0$, in which case the solution is zero as $t \to \infty$.

■

Hopefully, after walking through these the visuals the concepts are coming together.

1.4 Analyzing Equations Numerically

This section introduces you to two techniques for approximating solutions to a first order differential equation numerically. The first is taught in Calculus, Euler's Method, so it may be familiar. Euler's Method is not very accurate, but it is simple. The second is the most commonly used method,

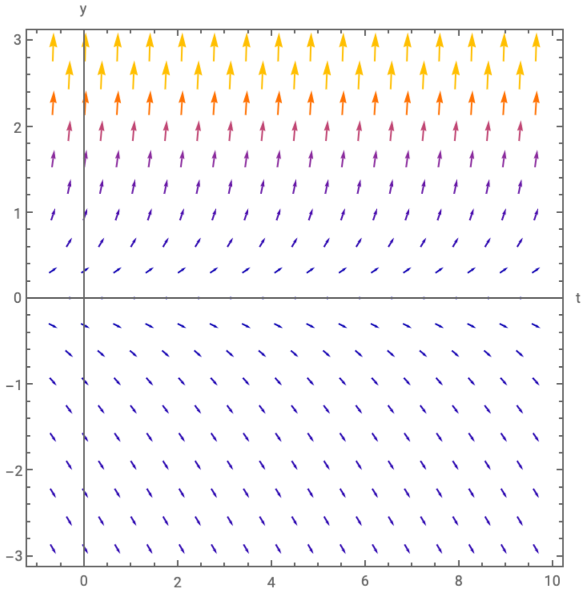


Figure 1.9: This is a direction field plot generated by Mathematica for $f(t,y) = e^y - 1$. When sketching solution curves, you will draw them by hand into the direction field plots. For each of the initial values given, find the point on the direction field graph, and then follow the arrows to sketch in the solution curve.
[add solution curve sketches]

Runge-Kutta 4. This method is used a lot, but if you want a full derivation and explanation please consider taking Numerical Analysis II. In this course, we will focus on its implementation and comparison with analytical solutions.

1.4.1 Euler's Method

Euler's method is a numerical solution technique that uses the slope value to approximate nearby values of $y(t)$. If we know $y' = f(t,y)$, we can use the slope and initial value to approximate the next point. You likely learned this technique, briefly, in Calculus II. It approximates the solution using time steps along the tangent line to the curve at each point.

We can discuss it in the context of the Taylor series expansion, $y(t_i + h) \approx y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi)$.

Noticing that $y'(t) = f(t,y)$, we can construct Euler's method:
$y(t_i + h) \approx y(t_i) + hf(t_i, y_i)$.
Where $h =$ step size, and $f(t_i, y_i) =$ slope of y at t_i.

■ **Example 1.19** Given $y' = 1 + t$, $y(0) = 1$. Determine $y(1)$ using $h = 0.25$ and Euler's Method.

First we use the initial condition as our starting point, and evaluate the derivative function $1 + t$ to determine the slope that we are stepping along. $y(0) = 1$, $y'(0) = 1 + 0 = 1$

Second, we substitute those values into Euler's method to calculate one step forward in the approximation. $y(0.25) = y(0) + h * f(0,1) = 1 + 0.25 * 1 = 1.25$

Repeat until we reach the endpoint desired. $y(0.5) = y(0.25) + h * f(.25, 1.25) = 1.25 + .25 * 1.25 = \frac{25}{16}$

$y(0.75) = y(0.5) + h * f(0.5, \frac{25}{16}) = \frac{25}{16} + \frac{3}{8} = \frac{31}{16}$

$y(1) = y(0.75) + h * f(0.75, \frac{31}{16}) = \frac{31}{16} + \frac{7}{16} = \frac{19}{8} = 2.375$ So, our approximate value for $y(1)$ is 2.375, using Euler's Method with a step size of $h = 0.25$.

If we solve for the analytical solution to the differential equation:

$\int dy = \int (1+t)dt$

$y = t + \frac{t^2}{2} + C$, then apply the initial condition $y(0) = 1$, we find that $C = 1$.

$y = t + \frac{t^2}{2} + 1$

Then $y(1) = 2.5$, actually.

Our error is given by $|2.5 - 2.375| = 0.125$, which is only 5% error.

This is the simplest technique, but it is not generally very accurate. It is used for its simplicity and efficiency.

If we double $h = 0.5$ and approximate $y(1)$, we have:

$y(0.5) = y(0) + 0.5 * f(0,1) = 1 + 0.5 = 1.5$

$y(1) = y(0.5) + 0.5 * f(0.5, 1.5) = 1.5 + 0.75 = 2.25$

The error is doubled also, to 0.25, or 10%. ■

This example corresponds with the linear convergence of Euler's method as $h \to 0$, the error of the method $\to 0$ linearly.

Local error: error incurred at each step.

Global error: total error in the approximation.

Let's do an example where Euler's Method is less accurate.

■ **Example 1.20** Approximate $y(1)$ for $y'(t) = 2y$, starting at the initial value $y(0) = 1$. Compare results using $h = 1$ and $h = 0.25$.

For $h = 1$, this approximation is simply $y(1) \approx y(0) + y'(0)h = 1 + 2 * 1 = 3$.

This is not likely to be a good approximation because taking a step of $h = 1$ is rather large. Euler's method becomes useful when we take smaller steps to reach the point we seek.

To determine $y(1)$ using $h = 0.25$ we do the same calculation, but we have to complete it 4 times.

$y(1/4) \approx y(0) + y'(0)h = 1 + 2 * 1/4 = 3/2$

Then, we use this approximation to build the next, and so forth.

$y(1/2) \approx y(1/4) + y'(1/4)h = 3/2 + 3 * 1/4 = 9/4$

$y(3/4) \approx y(1/2) + y'(1/2)h = 9/4 + 9/2(1/4) = 27/8$

$y(1) \approx y(3/4) + y'(3/4)h = 27/8 + 27/4(1/4) = 81/16 = 5.0625$

We can solve for the actual function:

$y' = 2y \quad \to \quad \frac{dy}{y} = 2dt \quad \to \quad \int \frac{dy}{y} = \int 2dt$

$\ln|y| = 2t + C \quad \to \quad y(t) = ke^{2t}$ where $k = e^C$.

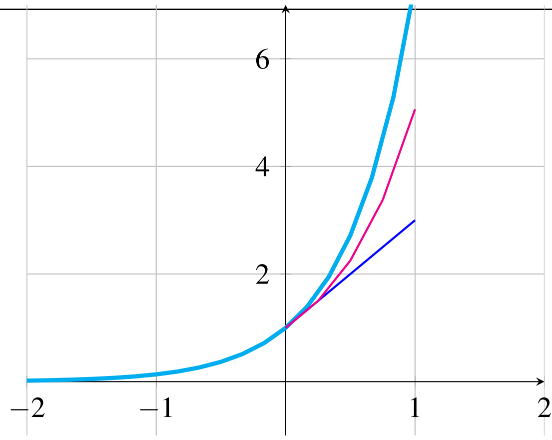
Using the initial value $y(0) = 1$, the solution is $y(t) = e^{2t}$, so $y(1) \approx 7.389$.

From this you can see that the approximation does get better with smaller steps, but it's still bad. For better accuracy, we will need to try even smaller steps than $1/4$.

Our absolute error for one step was $|7.389 - 3| = 4.389$ Huge! 59.4% error.
Our absolute error for four steps was $|7.389 - 5.0625| = 2.3265$ Huge, but better, 31.5% error.
Due to the large step size, we do not see the linear convergence clearly. With more approximations to compare it will approach linear convergence as $h \to 0$. ■

The approximation using Euler's Method is not going to be great, but it is handy for problems we cannot solve exactly. It also takes a <u>lot</u> of work to get decent accuracy.

The last example is particularly poor because e^{2t} is a very steep function. This is better seen with a visual: see the graph below of the function, and our two approximations.



The cyan curve is the actual function, the blue line is the approximation using only one step, and the magenta curve is the approximation using 4 steps.

We are trying to approximate the value of $y(1)$ without knowing $y(t)$, and only knowing $y'(t)$. This is why we use the tangent line at each point to propagate the approximation to the curve. For more details and analysis - please take Numerical Analysis II!

1.4.2 Runge-Kutta Methods

Naturally, many alternatives to Euler's Method have been developed to improve on its accuracy. There are multiple Runge-Kutta methods, but we will focus on RK4. These techniques are built off of numerical integration in Calculus II. Recall that we are attempting to approximate values of $y(t)$ for a a differential equation of the form $y'(t) = f(t,y)$.

Using the fundamental theorem of calculus, we can represent the solution through $y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(s,y(s))ds$. In the text, they mention approximations built on derivatives of $f(t,y)$ using a Taylor series' expansion. However, this is a riskier technique that becomes unstable quickly (for small h). Numerical derivatives divide by h, whereas numerical integration multiplies by h. Dividing by small numbers induces roundoff error, and is avoided when possible. Runge-Kutta methods numerically evaluate the integral $\int_{t_0}^{t_1} f(s,y(s))ds$, RK2 uses a trapezoidal rule and RK4 uses Simpson's rule. There is some intense algebra to derive the method, but at the end we define each step through

$$y(t_i+h) = y(t_i) + \frac{1}{6}(Y_{1,i} + 2Y_{2,i} + 2Y_{3,i} + Y_{4,i})$$

The $Y_{n,i}$ terms are representing approximations for the "next step." We take a linear combination of those approximations, which is weighted by the terms in Simpson's method, and that is used to make a more accurate step. Specifically, those approximations are given by

$$Y_{1,i} = hf(t_i, y(t_i))$$
$$Y_{2,i} = hf\left(t_i + \frac{h}{2}, y(t_i) + \frac{1}{2}Y_{1,i}\right)$$
$$Y_{3,i} = hf\left(t_i + \frac{h}{2}, y(t_i) + \frac{1}{2}Y_{2,i}\right)$$
$$Y_{4,i} = hf\left(t_i + h, y(t_i) + Y_{3,i}\right)$$

Note that the i's in these terms refer to the value of t that is used to construct them. We use i in numerical analysis to identify a step number. So, to construct the first step we reference the zeroth step (initial value), for the second step we reference values from the first, etc.

In this course, our emphasis will be on applying the method. For further analysis I recommend taking Numerical Analysis II!

To compare directly with Euler's method, we will use the examples from the last section, with RK4.

Example 1.21 Given $y' = 1+t$, $y(0) = 1$. Determine $y(1)$ using $h = 0.25$ and the RK4 Method.

First we use the initial condition as our starting point, and evaluate the derivative function $f(t,y) = 1+t$ to determine the terms needed for RK4.

$Y_{1,0} = hf(t_0, y(t_0)) = 0.25f(0,1) = 0.25(1+0) = 0.25$ or $1/4$

$Y_{2,0} = hf\left(t_0 + \frac{h}{2}, y(t_0) + \frac{1}{2}Y_{1,0}\right)$
$= 0.25f(0+0.125, 1+0.5*0.25)$
$= 0.25(1+0.125) = 9/32$

$Y_{3,0} = hf\left(t_0 + \frac{h}{2}, y(t_0) + \frac{1}{2}Y_{2,0}\right)$
$= 0.25f(0+0.125, 1+0.5*9/32)$
$= 0.25*(1+0.125) = 9/32$

$Y_{4,0} = hf(t_0+h, y(t_0)+Y_{3,0})$
$= 0.25f(0+0.25, 1+9/32)$
$= 0.25*(1+0.25) = 5/16$

We then combine them as prescribed by the technique:

$y(0.25) = y(0) + \frac{1}{6}(Y_{1,0} + 2Y_{2,0} + 2Y_{3,0} + Y_{4,0})$
$= 1 + \frac{1}{6}\left(\frac{1}{4} + \frac{2*9}{32} + \frac{2*9}{32} + \frac{5}{16}\right)$
$= 1 + \frac{1}{6} * \frac{27}{16} =$
$1 + \frac{9}{32} = \frac{41}{32} = 1.28125$

We complete this process three more times! You are not done yet.

$Y_{1,1} = hf(t_1, y(t_1)) = 0.25f(0.25, 41/32) = 0.25(1+0.25) = 5/16$

$Y_{2,1} = hf\left(t_1 + \frac{h}{2}, y(t_1) + \frac{1}{2}Y_{1,1}\right)$
$= 0.25f(0.25+0.125, 41/32+0.5*5/16)$
$= 0.25(1+0.375) = 11/32$

$Y_{3,1} = hf\left(t_1 + \frac{h}{2}, y(t_1) + \frac{1}{2}Y_{2,1}\right)$
$= 0.25f(0.25+0.125, 41/32+0.5*11/32)$
$= 0.25*(1+0.375) = 11/32$

$Y_{4,1} = hf(t_1+h, y(t_1)+Y_{3,1})$
$= 0.25f(0.25+0.25, 41/32+11/32)$
$= 0.25*(1+0.5) = 3/8$

$y(0.5) = y(0.25) + \frac{1}{6}(Y_{1,1} + 2Y_{2,1} + 2Y_{3,1} + Y_{4,1})$
$= \frac{41}{32} + \frac{1}{6}\left(\frac{5}{16} + \frac{2*11}{32} + \frac{2*11}{32} + \frac{3}{8}\right)$
$= \frac{41}{32} + \frac{1}{6} * \frac{33}{16}$
$= \frac{41}{32} + \frac{11}{32} = \frac{52}{32} = 1.625$

Two more!

$Y_{1,2} = hf(t_2, y(t_2)) = 0.25f(0.5, 52/32) = 0.25(1+0.5) = 3/8$

$Y_{2,2} = hf\left(t_2 + \frac{h}{2}, y(t_2) + \frac{1}{2}Y_{1,2}\right)$
$= 0.25f(0.5+0.125, 52/32+0.5*3/8)$
$= 0.25(1+0.625) = 13/32$

$Y_{3,2} = hf\left(t_2 + \frac{h}{2}, y(t_2) + \frac{1}{2}Y_{2,2}\right)$
$= 0.25f(0.5+0.125, 52/32+0.5*13/32)$
$= 0.25*(1+0.625) = 13/32$
$Y_{4,2} = hf(t_2+h, y(t_2)+Y_{3,2})$
$= 0.25f(0.5+0.25, 52/32+13/32)$
$= 0.25*(1+0.75) = 7/16$
$y(0.75) = y(0.5) + \frac{1}{6}(Y_{1,2}+2Y_{2,2}+2Y_{3,2}+Y_{4,2})$
$= \frac{52}{32} + \frac{1}{6}\left(\frac{3}{8} + \frac{2*13}{32} + \frac{2*13}{32} + \frac{7}{16}\right)$
$= \frac{52}{32} + \frac{1}{6} * \frac{39}{16}$
$= \frac{52}{32} + \frac{13}{32} = \frac{65}{32} = 2.03125$

Last time!

$Y_{1,3} = hf(t_3, y(t_3)) = 0.25f(0.75, 65/32) = 0.25(1+0.75) = 7/16$
$Y_{2,3} = hf\left(t_3 + \frac{h}{2}, y(t_3) + \frac{1}{2}Y_{1,3}\right)$
$= 0.25f(0.75+0.125, 65/32+0.5*7/16)$
$= 0.25(1+0.875) = 15/32$
$Y_{3,3} = hf\left(t_3 + \frac{h}{2}, y(t_3) + \frac{1}{2}Y_{2,3}\right)$
$= 0.25f(0.75+0.125, 65/32+0.5*15/32)$
$= 0.25*(1+0.875) = 15/32$
$Y_{4,3} = hf(t_3+h, y(t_3)+Y_{3,3})$
$= 0.25f(0.75+0.25, 65/32+15/32)$
$= 0.25*(1+1) = 1/2$
$y(1) = y(0.75) + \frac{1}{6}(Y_{1,3}+2Y_{2,3}+2Y_{3,3}+Y_{4,3})$
$= \frac{65}{32} + \frac{1}{6}\left(\frac{7}{16} + \frac{2*15}{32} + \frac{2*15}{32} + \frac{1}{2}\right)$
$= \frac{65}{32} + \frac{1}{6}\frac{45}{16}$
$= \frac{65}{32} + \frac{15}{32} = \frac{80}{32} = 2.5$

Comparison: Euler's approximation was 2.375, and the actual was 2.5.

Using RK4, our approximation is indistinguishable from the actual value! RK4 is an excellent approximation - it's worth the work. ∎

∎ **Example 1.22** Approximate $y(1)$ for $y'(t) = 2y$, starting at the initial value $y(0) = 1$. Compare results using $h = 1$ and $h = 0.25$.

First we use the initial condition as our starting point, and evaluate the derivative function $f(t,y) = 2y$ to determine the terms needed for RK4.

For $h = 1$, there is only one step needed.

$Y_{1,0} = hf(t_0, y(t_0)) = 1*f(0,1) = 1*(2*1) = 2$
$Y_{2,0} = hf\left(t_0 + \frac{h}{2}, y(t_0) + \frac{1}{2}Y_{1,0}\right)$
$= 1*f(0+0.5, 1+0.5*2)$
$= 1*(2*2) = 4$
$Y_{3,0} = hf\left(t_0 + \frac{h}{2}, y(t_0) + \frac{1}{2}Y_{2,0}\right)$

$= 1 * f(0 + 0.5, 1 + 0.5 * 4)$
$= 1 * (2 * 3) = 6$

$Y_{4,0} = hf(t_0 + h, y(t_0) + Y_{3,0})$
$= 1 * f(0 + 1, 1 + 6)$
$= 1 * (2 * 7) = 14$

$y(1) = y(0) + \frac{1}{6}(Y_{1,0} + 2Y_{2,0} + 2Y_{3,0} + Y_{4,0})$
$= 1 + \frac{1}{6}(2 + 2 * 4 + 2 * 6 + 14)$
$= 1 + \frac{36}{6} = 1 + 6 = 7.$

Comparison: Using Euler's method with one step we got an approximation of 3. The actual value was $e^2 \approx 7.389056$. So, we can see that RK4 is much closer to the actual value than Euler's method. Euler's method was nearly useless for this differential equation in one step, RK4 is reasonable.

The error in this approximation is $|7.389056 - 7| = 0.389056$, or 5.27%.

For $h = 0.25$, we'll need to do this work for 4 steps.

$Y_{1,0} = hf(t_0, y(t_0)) = 0.25 * f(0, 1) = 0.25 * (2 * 1) = 0.5$ or $1/2$

$Y_{2,0} = hf\left(t_0 + \frac{h}{2}, y(t_0) + \frac{1}{2}Y_{1,0}\right)$
$= 0.25 * f(0 + 0.125, 1 + 0.5 * 0.5)$
$= 0.25 * (2 * 1.25) = 10/16$

$Y_{3,0} = hf\left(t_0 + \frac{h}{2}, y(t_0) + \frac{1}{2}Y_{2,0}\right)$
$= 0.25 * f(0 + 0.125, 1 + 0.5 * 10/16) = 0.25 * (2 * 42/32) = 42/64$

$Y_{4,0} = hf(t_0 + h, y(t_0) + Y_{3,0})$
$= 0.25 * f(0 + 0.25, 1 + 42/64) = 0.25 * (2 * 106/64) = 106/128$

$y(0.25) = y(0) + \frac{1}{6}(Y_{1,0} + 2Y_{2,0} + 2Y_{3,0} + Y_{4,0})$
$= 1 + \frac{1}{6}\left(\frac{1}{2} + 2 * \frac{10}{16} + 2 * \frac{42}{64} + \frac{106}{128}\right)$
$= 1 + \frac{249}{384} = \frac{633}{384} = \frac{211}{128} \approx 1.648438.$

We complete this process three more times! You are not done yet.

$Y_{1,1} = hf(t_1, y(t_1)) = 0.25 * f(0.25, 211/128) = 0.25 * (2 * 211/128) = 211/256$

$Y_{2,1} = hf\left(t_1 + \frac{h}{2}, y(t_1) + \frac{1}{2}Y_{1,1}\right)$
$= 0.25 * f(0.25 + 0.125, 211/128 + 0.5 * 211/256) = 0.25 * (2 * 1055/512) = 1055/1024$

$Y_{3,1} = hf\left(t_1 + \frac{h}{2}, y(t_1) + \frac{1}{2}Y_{2,1}\right)$
$= 0.25 * f(0.25 + 0.125, 211/128 + 0.5 * 1055/1024) = 0.25 * (2 * 4431/2048) = 4431/4096$

$Y_{4,1} = hf(t_1 + h, y(t_1) + Y_{3,1})$
$= 0.25 * f(0.25 + 0.25, 211/128 + 4431/4096) = 0.25 * (2 * 11183/4096) = 11183/8192$

$y(0.5) = y(0.25) + \frac{1}{6}(Y_{1,1} + 2Y_{2,1} + 2Y_{3,1} + Y_{4,1})$
$= \frac{211}{128} + \frac{1}{6}\left(\frac{211}{256} + 2 * \frac{1055}{1024} + 2 * \frac{4431}{4096} + \frac{11183}{8192}\right)$
$= \frac{211}{128} + \frac{1}{6}\left(\frac{6752}{8192} + \frac{16880}{8192} + \frac{17724}{8192} + \frac{11183}{8192}\right)$
$= \frac{211}{128} + \frac{1}{6}\frac{52539}{8192}$

$= \dfrac{27008}{16384} + \dfrac{17513}{16384} = \dfrac{44521}{16384} \approx 2.717346.$

Two more!

$Y_{1,2} = hf(t_2, y(t_2)) = 0.25f(0.5, 44521/16384) = 0.25(2*44521/16384) = 44521/32768$

$Y_{2,2} = hf\left(t_2 + \dfrac{h}{2}, y(t_2) + \dfrac{1}{2}Y_{1,2}\right)$

$= 0.25f(0.5+0.125, 44521/16384 + 0.5*44521/32768) = 0.25(2*222605/65536) = 222605/131072$

$Y_{3,2} = hf\left(t_2 + \dfrac{h}{2}, y(t_2) + \dfrac{1}{2}Y_{2,2}\right)$

$= 0.25f(0.5+0.125, 44521/16384 + 0.5*222605/131072) = 0.25*(2*934941/262144) = 934941/524288$

$Y_{4,2} = hf(t_2 + h, y(t_2) + Y_{3,2})$

$= 0.25f(0.5+0.25, 44521/16384 + 934941/524288) = 0.25*(2*2359613/524288) = 2359613/1048576$

$y(0.75) = y(0.5) + \dfrac{1}{6}(Y_{1,2} + 2Y_{2,2} + 2Y_{3,2} + Y_{4,2})$

$= \dfrac{44521}{16384} + \dfrac{1}{6}\left(\dfrac{44521}{32768} + 2*\dfrac{222605}{131072} + 2*\dfrac{934941}{524288} + \dfrac{2359613}{1048576}\right)$

$= \dfrac{44521}{16384} + \dfrac{1}{6}*\left(\dfrac{1424672}{1048576} + \dfrac{3561680}{1048576} + \dfrac{3739764}{1048576} + \dfrac{2359613}{1048576}\right)$

$= \dfrac{5698688}{2097152} + \dfrac{3695243}{2097152} = \dfrac{9393931}{2097152} \approx 4.479375$

Last time!

$Y_{1,3} = hf(t_3, y(t_3)) = 0.25f(0.75, 9393931/2097152) = 0.25(2*9393931/2097152) = 9393931/4194304$

$Y_{2,3} = hf\left(t_3 + \dfrac{h}{2}, y(t_3) + \dfrac{1}{2}Y_{1,3}\right)$

$= 0.25f(0.75+0.125, 9393931/2097152 + 0.5*9393931/4194304) = 0.25(2*46969655/8388608) = 46969655/16777216$

$Y_{3,3} = hf\left(t_3 + \dfrac{h}{2}, y(t_3) + \dfrac{1}{2}Y_{2,3}\right)$

$= 0.25f(0.75+0.125, 9393931/2097152 + 0.5*46969655/16777216) = 0.25*(2*197272551/33554432) = 197272551/67108864$

$Y_{4,3} = hf(t_3 + h, y(t_3) + Y_{3,3})$

$= 0.25f(0.75+0.25, 9393931/2097152 + 197272551/67108864) = 0.25*(2*497878343/67108864) = 497878343/134217728$

$y(1) = y(0.75) + \dfrac{1}{6}(Y_{1,3} + 2Y_{2,3} + 2Y_{3,3} + Y_{4,3})$

$= \dfrac{9393931}{2097152} + \dfrac{1}{6}\left(\dfrac{9393931}{4194304} + 2*\dfrac{46969655}{16777216} + 2*\dfrac{197272551}{67108864} + \dfrac{497878343}{134217728}\right)$

$= \dfrac{9393931}{2097152} + \dfrac{1}{6}\left(\dfrac{300605792}{134217728} + \dfrac{751514480}{134217728} + \dfrac{789090204}{134217728} + \dfrac{497878343}{134217728}\right)$

$= \dfrac{1202423168}{268435456} + \dfrac{779696273}{268435456} = \dfrac{1982119441}{268435456} \approx 7.383970$

Comparison: Euler's approximation was 5.0625, and the actual was $e^2 \approx 7.389056$.

Significantly improved! Still not great, but h is still quite large (to help the example fit on the board).

Our error using Euler was $|7.389056 - 5.0625| = 2.326556$, or 31.49%. Euler has linear convergence, which means that the error is some order of the value of h.

Our error using RK4 is $|7.389056 - 7.383970| = 0.005086$, or 0.07%. RK4 has order h^4 convergence, which is significantly better than Euler. ■

Error for an s-stage RK method is $\mathcal{O}(h^s)$ generally.

Implementing RK methods: they are simple and flexible, but they can be computationally expensive due to all the sub-steps. Deriving these methods is very long and complex (which is why we emphasize this in Numerical Analysis rather than Differential Equations.)

You can confirm the error by plotting your error with respect to h on a loglog scale, and the slope will correspond to the power of h in the accuracy.

$error = ch^q$ Take a natural logarithm of both sides.

$\log(error) = \log(c) + q\log(h)$ Linear relationship between $\log(error)$ and $\log(h)$.

Slope of the line is q. Given two values, we can approximate the line. $q \approx \dfrac{\log(e_1) - \log(e_2)}{\log(h_1) - \log(h_2)}$.

For our last example, $q \approx \dfrac{\log(0.389056) - \log(0.005086)}{\log(1) - \log(0.25)} \approx 3.12865$ (using $h = 1$ is always a poor approximation, so it is a poor comparison to approximate q, but we're close to 4. For small values of h this should be close to 4. When I use the error from $h = 0.25$ with $h = 0.1$, the approximation to q is 3.7292, for example.)

You will notice that higher order methods are better when you require <u>high</u> accuracy.

1.5 First Order Linear Differential Equations

This section outlines the remaining form of first order linear equations. We will focus on solving differential equations of the general form $y' + p(t)y = q(t)$ and applications that have this form.

1.5.1 Integrating Factor

The primary solution technique for nonseparable first order linear differential equations is the integrating factor method. This technique creates a product rule with $y(t)$ and an integrating factor $\mu(t)$. The integrating factor is a function of t which allows us to rewrite the equation as a derivative of $\mu(t)y(t)$. Using a product rule, $(\mu(t)y(t))' = \mu'(t)y(t) + \mu(t)y'(t)$, which takes a similar form to our general form above.

Exponential functions are unique because the function remains when we apply a derivative, $\dfrac{d}{dt}\left[e^{kt}\right] = ke^{kt}$. We exploit this form in defining the integrating factor. Our integrating factor $\mu(t) = e^{\int p(t)dt}$ ensures that $p(t)$ is multiplied by $y(t)$ when the derivative is applied.

$\dfrac{d}{dt}\left[e^{\int p(t)dt}y(t)\right] = p(t)e^{\int p(t)dt}y(t) + e^{\int p(t)dt}y'(t) = e^{\int p(t)dt}\left(p(t)y + y'\right)$ (the form of our left side in the general form $y' + p(t)y = q(t)$.)

In order to apply the integrating factor technique, we first define $\mu(t) = e^{\int p(t)dt}$, then multiply the entire equation by the integrating factor (this ensures that we do not change the underlying relationships.)

$y'e^{\int p(t)dt} + p(t)ye^{\int p(t)dt} = q(t)e^{\int p(t)dt}$

We then compress the left side to represent a single derivative

$\left(y(t)e^{\int p(t)dt}\right)' = q(t)e^{\int p(t)dt}$

Then integrate both sides of the equation with respect to t

$y(t)e^{\int p(t)dt} = \int q(t)e^{\int p(t)dt}dt$

and solve for $y(t) = e^{-\int p(t)dt}\int q(t)e^{\int p(t)dt}dt$

■ **Example 1.23** Solve the first order linear differential equation $y' + 2y = t$ for $y(t)$.

First, we recognize the pieces in the equation. This is already in the general form, so $p(t) = 2$ and $q(t) = t$.

Define the integrating factor $\mu(t) = e^{\int 2dt} = e^{2t}$.

Multiply both sides by the integrating factor $y'e^{2t} + 2ye^{2t} = te^{2t}$.

Compress the left side of the equation to represent it as a single derivative $\left(ye^{2t}\right)' = te^{2t}$.

Integrate both sides with respect to t: $y(t)e^{2t} = \int te^{2t}dt$

The integral $\int te^{2t}dt$ requires integration by parts. $u = t$, $dv = e^{2t}dt$.

$\int te^{2t}dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$ (we again write our "+C" on the right side.)

Then $y(t)e^{2t} = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$.

Solving for $y(t) = \frac{1}{2}t - \frac{1}{4} + Ce^{-2t}$, the general solution of our first order linear differential equation.

We can verify that it works by substituting y and y' into the original equation. $y'(t) = \frac{1}{2} - 2Ce^{-2t}$

$y' + 2y = \frac{1}{2} - 2Ce^{-2t} + 2\left(\frac{1}{2}t - \frac{1}{4} + Ce^{-2t}\right) = \frac{1}{2} - 2Ce^{-2t} + t - \frac{1}{2} + 2Ce^{-2t} = t$ ✓Consistent!

■

This will also hold for more complicated functions in the place of $p(t)$ and $q(t)$.

■ **Example 1.24** Solve the first order linear differential equation $t^2y' - 4y = 1$ for $y(t)$.

This equation will require algebraic manipulation to convert it to the general form. $y' - \frac{4}{t^2}y = \frac{1}{t^2}$.

Dividing by t^2 causes issues with $t = 0$, so there will be constraints on the solution: $t \neq 0$.

Define the integrating factor $\mu(t) = e^{\int -4dt/t^2} = e^{4/t}$

Multiply both sides by the integrating factor $y'e^{4/t} - \frac{4}{t^2}ye^{4/t} = \frac{1}{t^2}e^{4/t}$

Compress the left side of the equation to a single derivative $\left(y(t)e^{4/t}\right)' = \frac{1}{t^2}e^{4/t}$

Integrate both sides with respect to t: $y(t)e^{4/t} = \int \frac{1}{t^2}e^{4/t}dt$

This integral requires a u-substitution. $u = \frac{4}{t}$, $du = \frac{-4}{t^2}dt$.

$y(t)e^{4/t} = \frac{-1}{4}e^{4/t} + C$

Solve for $y(t) = \frac{-1}{4} + Ce^{-4/t}$.

For a reassurance, we can verify the solution: $t^2y' - 4y = t^2\left(\frac{4}{t^2}Ce^{-4/t}\right) - 4\left(\frac{-1}{4} + Ce^{-4/t}\right) = 4Ce^{-4t} + 1 - 4Ce^{-4/t} = 1$ ✓

■

This example helps us start the conversation about existence and uniqueness.

Application Problems

There are a few models which use this form of equation we will discuss: Newton's Law of Cooling and Mixing Problems.

Newton's Law of Cooling Newton's Law of Cooling is a model for the change in temperature of an object dependent upon the temperature of its surroundings. The simplest form of this model is $\frac{dT}{dt} = k(E - T)$, where E is the temperature of the evironment, it can be a constant or a function of t. This model is the foundation for many temperature models, most of the examples in textbooks use a hot cup of coffee cooling on a counter, or the temperature of a dead body. However, it extends more broadly to temperature variation in materials, like in buildings or bridges. At a higher level, it's improved upon to model heat transfer.

The model is not in the general form for our first order linear differential equations, but is easily converted to the general form we've been using in this section.

$T' + kT = kE$

In this equation, $p(t) = k$ and $q(t) = kE(t)$. So, it is solved using integrating factor technique.

The solution is of the form $T(t) = e^{-kt}\int kE(t)e^{kt}dt$

When $E(t)$ is a constant, it simplifies to the form we use in algebra: $T(t) = E + Ce^{-kt}$, or $T(t) = E + (T_0 - E)e^{-kt}$ for an initial condition $T(0) = T_0$.

We can verify this form of solution through $T'(t) = -k(T_0 - E)e^{-kt} = k(E - T_0)e^{-kt}$ and see that when the surrounding temperature (E) is greater than the initial temperature of the object (T_0), the temperature (T) will increase ($T' > 0$) as it approaches the surrounding temperature. Likewise, if the surrounding temperature (E) is less than the initial temperature of the object (T_0), then the temperature (T) will decrease ($T' < 0$) as it approaches the surrounding temperature. This is as we would expect; if you set an ice cube on your counter, its temperature will increase and it will melt. If you set a cup of tea on your counter, its temperature will decrease (and you may be disappointed if it gets "too cold.")

Mixing Problems Mixing problems are used predominantly to model the contamination of bodies of water. However, it doesn't have to model a contaminant. Sometimes we'll use these models when we're looking at adding chlorine to a pool, or we'll use it to model the filtration of water (removal of a contaminant), it all depends on the problem. The version we use in this course is based on the concept that the rate of change in the contaminant is directly proportional to the amount moving into the body of water subtracting the amount leaving the body of water, $\dfrac{dQ}{dt} = Q_{in} - Q_{out}$. Some of the assumptions used for this model include: homogenous mixing. When the contaminant enters the water it is promptly dispersed evenly through the body of water. In practice, this takes time, but in an ideal model that we can solve - it serves our purposes.

To define Q_{in}, there has to be a defined source of contamination with a given flow rate r_{in} and concentration C_{in}. Using that information, we can then define the mass of contaminant coming into the body of water through $C_{in}r_{in}$ Concentration is a unit of mass per unit volume, and flow rate is a measure of volume per time. When we do some baby dimensional analysis we see that the units become mass per unit time, which is consistent with the definition of $\dfrac{dQ}{dt}$ where Q is the mass of the contaminant in our body of water.

To define Q_{out}, there needs to be some amount of the body of water leaving (a leak, a stream, etc.) We then have a concentration in our body of water that is given by the current mass of contaminant Q divided by the total volume of the given body of water V. If V is changing, the problem must compensate by making it a function, but if V is fixed we treat it as a constant. Then, Q_{out} is defined in the same manner as Q_{in}, the concentration in the body of water $C_{out} = Q/V$ and the flow rate of the water that is leaving the body of water r_{out}.

Combining these, we form the equation $\dfrac{dQ}{dt} = C_{in}r_{in} - \dfrac{Qr_{out}}{V}$. This is another first order linear differential equation, and it is of the form $y' + p(t)y = q(t)$:

$$Q' + \frac{r_{out}}{V(t)}Q = C_{in}r_{in}$$

If you have any interest in fluid dynamics (which involves much more complicated DEs) or water quality models, this is the foundation for understanding the transport of contents in water. I personally know folks who work with these models in their jobs (they get more complicated than this, but the foundation is there.)

So, let's do an example of a filtration problem and then another example with a contaminant leaking into a lake.

■ **Example 1.25** In a water filtration process, the water is pushed through a filter that reduces the concentration of contaminants by 70% of the initial concentration. As the water is pushed through the filter, the flow rates are equal because the overall body of water does not change. For a body of water that is $22,000$ gallons, with the flow rate through the filter as 4 gallons per minute: represent this system as a differential equation and then solve for the general solution.

When we construct the equation, remember that the input is the initial concentration, and the output is the reduced concentration by the filter. The flow rates are equal, $r_{in} = r_{out}$, so we will remove the subscripts and refer to the flow rate by r alone.

$$\frac{dQ}{dt} = r\left(\frac{0.7Q}{22000} - \frac{Q}{22000}\right) = 4\left(\frac{-0.3Q}{22000}\right) = -0.00005455Q$$

This equation is a simplified version of the general model because there is no added contaminants, only contaminants that are removed. The differential equation is then a separable equation, and we can solve it using a separation of variables.

$$\frac{dQ}{Q} = -0.00005455dt$$

Integrating both sides yields $\ln|Q| = -0.00005455t + C$

Solving for $Q(t) = ke^{-0.00005455t}$ where $k = e^C$.

The general solution shows us that the contaminants are reduced at a very slow rate. This is likely due to the facts that 1) not all contaminants are removed, and 2) the flow rate is very small relative to the total body of water. ■

■ **Example 1.26** *Big Chemical Corporation* is situated on the side of Lake Squirrel (somehow the EPA hasn't found them yet, it's not a popular lake.) A fresh environmental engineer spotted a leaky trail from their facility pouring into the lake. So naturally, they brought out some test equipment (discreetly) to take samples and measure the flow rate of the trail near the edge of the lake. Based on their analysis, the flow rate is approximately 1.5 gallons per minute, and when they evaluated the sample it contained a concentration of 3 grams per ounce of fluid (the engineer did not have access to a spectrometer, so the contaminant is unknown.)

There is a small creek pouring off the lake, down a hill. We will solve two independent versions of this problem. 1) assume the creek has the same flow rate as the leak coming into the lake.

2) assume the creek has a different flow rate, our engineer will measure it for us when we get to this version of the problem.

For both of these cases, we need an estimated volume for lake Squirrel. Our engineer has been trained how to estimate this based on measurements of the boundary and several depth measurements. So, they take a small boat out on Lake Squirrel, measure various depths, complete some calculations, and estimate the volume of the lake to be approximately 3.5×10^{10} gallons. (This value is based on the volume of Lake Carter in CO.)

We will also need to convert all given values to consistent units. 3 grams per ounce must also be converted to reference gallons. There are 128 ounces per gallon, so there are approximately 384 grams per gallon.

First, the simpler case: 1) assume the creek has the same flow rate as the leak coming into the lake. Determine a differential equation to model this problem, and then solve for its general solution.

We construct the equation so that $r_{in} = r_{out}$ so we will just use r (as in the last example). Keep in mind that Q is in grams, and t is in minutes based on the problem described.

$$\frac{dQ}{dt} = r\left(C_{in} - \frac{Q}{V}\right) = 1.5\left(384 - \frac{Q}{3.5 \times 10^{10}}\right)$$

In general form this becomes $Q' + \frac{1.5}{3.5 \times 10^{10}}Q = 576$

We apply an integrating factor to solve this equation $\mu(t) = e^{\int 4.286 \times 10^{-11} dt}$

$$\left(Qe^{4.286 \times 10^{-11}t}\right)' = 576e^{4.286 \times 10^{-11}t}$$

Integrate both sides with respect to t

$$Qe^{4.286 \times 10^{-11}t} = 1.344 \times 10^{13} e^{4.286 \times 10^{-11}t} + C$$

$$Q(t) = 1.344 \times 10^{13} + Ce^{-4.286 \times 10^{-11}t}$$

From this solution, we see that there is a rather small *change* in the contaminant from the leak. However, we also see that as $t \to \infty$, the mass of contaminant will approach 1.344×10^{13} grams. Using the volume of the lake to see what that contamination would look like in terms of concentration, it's 383.97 grams per gallon. What that means is that, if left alone, the concentration in the lake will approach the concentration from the leak (assuming nothing changes.) If we convert that to ppm (for relevancy), that is 101442ppm! For context, we want to keep ppm less than 100 in domestic waters. This concentration may change slowly, but it does not mean that the leak should be ignored.

The second case 2) Assume the creek has a different flow rate. The engineer measures a flow rate of 30 gallons per minute leaving the lake. This changes our problem significantly.

Now we have to model the change in volume first, and define $V(t)$.

$\frac{dV}{dt} = r_{in} - r_{out}$

$\frac{dV}{dt} = 1.5 - 30 = -28.5$

$V(t) = -28.5t + C$

The estimated current volume was 3.5×10^{10} gallons, so $C = 3.5 \times 10^{10}$

$V(t) = 3.5 \times 10^{10} - 28.5t$

Now we can use $V(t)$ in our equation from the first model. Nothing else changes (just r_{out} and $V(t)$.)

$\frac{dQ}{dt} = \left(r_{in}C_{in} - \frac{r_{out}Q}{V(t)}\right) = \left(1.5 * 384 - \frac{30Q}{3.5 \times 10^{10} - 28.5t}\right)$

Rewriting this equation in general form yields $Q' + \frac{30}{3.5 \times 10^{10} - 28.5t}Q = 576.$

We define a new integrating factor for the new equation $\mu(t) = e^{\int 30/(3.5\times 10^{10} - 28.5t)dt} = e^{-30/28.5 \ln|3.5\times 10^{10} - 28.5t|} = \left(3.5 \times 10^10 - 28.5t\right)^{-30/28.5}$.

$\left(Q(3.5 \times 10^{10} - 28.5t)^{-30/28.5}\right)' = 576\left(3.5 \times 10^{10} - 28.5t\right)^{-30/28.5}$.

Integrating both sides with respect to t (requires a u-substitution on the right)

$Q\left(3.5 \times 10^{10} - 28.5t\right)^{-30/28.5} = 384\left(3.5 \times 10^{10} - 28.5t\right)^{-1.5/28.5} + C$

$Q(t) = 384\left(3.5 \times 10^{10} - 28.5t\right) + C\left(3.5 \times 10^{10} - 28.5t\right)^{30/28.5}$

From this solution we see that out model is completely different. When the flow rate leaving the lake is greater than that into the lake, the volume is decreasing. When the volume is decreasing, the concentration increases more rapidly, even though the contaminant flowing into the lake remains the same. In this case, the physical scenario is quite different. Think about the implications of this situation, and then consider the opposite: what if the flow rate out of the lake were slower than the flow rate into the lake? ■

1.5.2 Exact Equations

The method of integrating factor introduces the form of equation we refer to as **Exact**. An exact equation is defined by a form related to what we did to the equations when we multiplied them by $\mu(t)$ - we made the left side a single derivative with respect to t. In an exact equation, the left side will be the same, but the right side is zero. So, while the form on the left resembles that of our integrating factor technique, the equation is only exact if the right side is zero. We use integrating factor technique to convert to an exact form.

Generally, we denote an exact equation of the form $\Psi_x(x,y) + \Psi_y(x,y)\frac{dy}{dx} = 0$ (recall from Calculus 3, this is saying that the divergence of Ψ is zero.) More generally we recognize exact equations in a form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, where $M_y(x,y) = N_x(x,y)$ (or $\Psi_{xy}(x,y) = \Psi_{yx}(x,y)$

using Clairaut's theorem in Calculus III.) These are a special case, wherein applying the integrating factor technique forms $\frac{d}{dt}[\Psi(x,y(x))] = 0$.

After integrating both sides with respect to t, we have a situation where $\Psi(x,y(x)) = C$ (an implicit solution to our ODE.)

■ **Example 1.27** Determine if the following equation is exact, then solve the equation $3x^2y+4+(x^3-2)y' = 0$, $y(0) = 5$.

The term in front of y' is $N(x,y) = x^3-2$ ($\Psi_y(x,y)$)
The term by itself is $M(x,y) = 3x^2y+4$ ($\Psi_x(x,y)$)
To check if the equation is exact, we apply Clairaut's theorem (Is $\Psi_{xy} = \Psi_{yx}$?)
$N_x = \Psi_{yx} = 3x^2$
$M_y = \Psi_{xy} = 3x^2$
Since these match, we know the equation is exact.

To solve the equation, we can apply integrating factor as-is, or we can use the fact that it is an exact equation to write it as $\frac{d}{dt}[\Psi(x,y(x))] = 0$.

So, what is $\Psi(x,y(x))$? We can use our integration techniques and practice from Calculus III to determine $\Psi(x,y)$ (called a potential function in Calculus III.)

We determine the antiderivatives of the two partial derivatives, and match the terms.
$\int N(x,y)dy = \int \left(x^3-2\right) dy = x^3y-2y+C(x)$
and
$\int M(x,y)dx = \int \left(3x^2y+4\right) dx = x^3y+4x+D(y)$

These two results must be consistent with each other. We recognize that x^3y is in both, which is good! $-2y$ only shows up on the first term because it is a term only of y, so when the derivative with respect to x was taken - it disappeared. The same is true of the $4x$ term, but opposite variables.

When we combine the pieces, we find that $\Psi(x,y) = x^3y+4x-2y+k$, where k is an additive constant.

So, when we integrate $\frac{d}{dx}[\Psi(x,y(x))] = 0$ we define an implicit relationship between x and y.
$x^3y+4x-2y+k = C_1$, combine those constants, then solve for $y(x)$ (if explicit form is needed).
$C_2 = C_1 - k$
$x^3y+4x-2y = C_2$
$(x^3-2)y+4x = C_2$
$(x^3-2)y = C_2-4x$
$y(x) = \dfrac{C_2-4x}{x^3-2}$
Initial value $y(0) = 5$,
$y(0) = \dfrac{C_2-4*0}{0^3-2} = \dfrac{C_2}{-2} = 5$
$C_2 = -10$
$y(x) = \dfrac{-10-4x}{x^3-2}$
■

Another common form of exact equations is written as $M(x,y)dx+N(x,y)dy = 0$. These equations are consistent, and there is no need to be concerned by the notation. When you divide by dx, it looks exactly the same $M(x,y)+N(x,y)\frac{dy}{dx} = 0$. Treat the problem in exactly the same way.

1.6 Existence and Uniqueness

Assurance of existence and uniqueness of solutions is important for solving differential equations. In this course, most differential equations we encounter have solutions that exist and are unique for the problem posed. However, when speaking more broadly about differential equations and when considering real-world problems - there are always the underlying questions: Does the solution to this problem exist? If it exists, is it unique?

First Order Linear Differential Equations

The following theorem is the first in the analysis of a differential equation for existence and uniqueness of its solution, and it is specifically for the general form used for a first order linear differential equation.

Theorem 1.6.1 If a differential equation of the form $y' + p(t)y = q(t)$ has an initial value, $y(t_0) = y_0$, and the functions $p(t)$ and $q(t)$ exist and are defined on an open interval (a,b), then there exists a unique function which satisfies the differential equation and initial condition.

In order to apply this theorem to a problem, we must first manipulate the equation to its general form $y' + p(t)y = q(t)$. Then, we evaluate the functions $p(t)$ and $q(t)$ to determine if the functions are continuous on an open interval. Recall: Continuity from Calculus I and III. If there are discontinuities, they must be excluded from the open interval to define a unique solution. If no initial condition is given, the solution will not be unique. The initial condition is what constrains the general solution (family of solutions) to a single solution.

■ **Example 1.28** Determine the existence and uniqueness of solutions to the previous example $t^2y' - 4y = 1$ with initial condition $y(1) = 2$.

When we manipulated the expression to general form: $y' + \dfrac{-4}{t^2}y = \dfrac{1}{t^2}$

$p(t) = \dfrac{-4}{t^2}$ has a discontinuity at $t = 0$

$q(t) = \dfrac{1}{t^2}$ also has a discontinuity at $t = 0$

Therefore, we cannot guarantee a unique solution on the open interval $(-\infty, \infty)$ (the interval for a case without any discontinuities.) However, we can remove a single point and form an open interval that excludes $t = 0$. When we remove $t = 0$, we are left with two open intervals $(-\infty, 0)$ and $(0, \infty)$.

We use the initial condition to define which open interval is useful for this problem $y(1) = 2$ is using $t = 1$ for the initial condition and $t = 1$ is in the interval $(0, \infty)$.

Based on the theorem for the existence and uniqueness of a solution for a first order linear differential equation of the form $y' + p(t)y = q(t)$, we can guarantee a solution exists and is unique for $t^2y' - 4y = 1$ with initial condition $y(1) = 2$ in the open interval $(0, \infty)$. ■

This process is the same for multiple discontinuities, we seek intervals that exclude all discontinuities and include the initial condition. If that cannot be defined, then we cannot guarantee existence and/or uniqueness.

First Order Differential Equations

This theorem expands the definition so that the differential equation no longer needs to be linear. The general form of such equations is $\dfrac{dy}{dt} = f(t,y)$.

Theorem 1.6.2 Given a first order differential equation $y' = f(t,y)$ with initial condition $y(t_0) = y_0$, if the functions $f(t,y)$ and $\dfrac{df}{dy}(t,y)$ are continuous functions on a defined rectangle $R = \{(t,y) | 0 < |t - t_0| < a, 0 < |y - y_0| < b\}$, then there exists a unique solution $y(t)$ for $y' = f(t,y)$ and $y(t_0) = y_0$ on some interval $|t - t_0| < h$ which is contained within the interval $|t - t_0| < a$.

In more words than math, this theorem is stating that when the function $f(t,y)$ and its derivative with respect to y are continuous on an open rectangle, R, there will exist a unique solution to the problem on a sub-interval of the t interval defining R (it can be on the t interval defining R, but it can also be a smaller interval).

We use this theorem to check if a given differential equation has a unique solution, much like we did in the last section.

■ **Example 1.29** Given $\frac{dy}{dt} = \frac{y}{t-1}$, $y(0) = 2$, determine if a unique solution exists to this problem using the theorem for first order differential equations.

First we must determine if the conditions are met. Is $f(t,y)$ continuous on an open rectangle? Is $\frac{df}{dy}$ continuous on an open rectangle? Can we define an open rectangle where both are continuous?

$f(t,y)$ is not continuous when $t = 1$, so as long as we exclude $t = 1$, we can define an open interval in t. There are no constraints on y. So, a rectangle for $f(t,y)$ is $R = \{(t,y)|0 < |t-0| < 1, 0 < |y-2| < \infty\}$.

$\frac{df}{dy} = \frac{1}{t-1}$ is also not continuous when $t = 1$, but is continuous for all values of y. Then, $\frac{df}{dy}$ is continuous on the same R as $f(t,y)$.

We can define an open region $R = \{(t,y)|0 < |t-0| < 1, 0 < |y-2| < \infty\}$. Therefore, there exists a unique solution to this differential equation on a subinterval of $|t| < 1$, or $(-1,1)$.

If we solve this differential equation using separation of variables,

$\ln|y| = \ln|t-1| + C$

$y(t) = k(t-1)$, but only when $|t| < 1$. ■

Bernoulli Equations

Bernoulli equations are an extension of first order equations. They are first order nonlinear equations that take a certain form.

$y' + P(t)y = Q(t)y^n$

Due to the similarity of form, we discuss solutions to Bernoulli equations immediately after integrating factor.

We determine a multiplier (like we used for integrating factor) for the equation based on the value of n. For this form, we multiply the entire equation by $(1-n)y^{-n}$.

The new equation becomes $(1-n)y^{-n}y' + P(t)(1-n)y^{1-n} = (1-n)Q(t)$.

If we recognize that the first term is the derivative of y^{1-n} with respect to t, $\frac{d}{dt}\left[y^{1-n}\right] = (1-n)y^{-n}\frac{dy}{dt}$, we can rewrite the equation to look like a linear differential equation (it's still nonlinear).

$\left(y^{1-n}\right)' + P(t)(1-n)\left(y^{1-n}\right) = (1-n)Q(t)$

If we were to rewrite it like a u-substitution problem, it becomes:

$u' + P(t)(1-n)u = (1-n)Q(t)$, which we can solve using integrating factor technique.

Then, after we've solved for $u(t)$, we can solve algebraically for y through the relation $u = y^{1-n}$.

■ **Example 1.30** Solve the Bernoulli equation $y' + ty = ty^3$ satisfying the initial condition $y(0) = 1$.

First, we multiply both sides by $(1-n)y^{-n}$. In this problem $n = 3$, so we will multiply by $-2y^{-3}$.

$-2y^{-3}y' - 2ty^{-2} = -2t$, then we rewrite $-2y^{-3}y'$ as $\frac{d}{dt}\left[y^{-2}\right]$. If we choose to do a u-substitution here, we can say $u(t) = y^{-2}$, and our equation becomes:

$u' - 2tu = -2t$. Then we solve this equation as we solved other first order linear equations.

Define an integrating factor $\mu(t) = e^{\int -2t\,dt} = e^{-t^2}$

Multiply both sides by the integrating factor

$e^{-t^2}u' - 2te^{-t^2}u = -2te^{-t^2}$
Compress the derivative on the left side (product rule)
$\left(e^{-t^2/2}u\right)' = -2te^{-t^2}$
Integrate both sides with respect to t (u-substitution required on the right side)
$e^{-t^2}u = e^{-t^2} + C$
Solve for u
$u(t) = 1 + Ce^{t^2}$
Solve for y
$y^{-2} = 1 + Ce^{t^2}$
$y(t) = \sqrt{\dfrac{1}{1+Ce^{t^2}}}$ (The implied $\pm$ is positive because our initial condition is positive)
$y(0) = \sqrt{\dfrac{1}{1+C}} = 1$ Solve for C
$C = 0$
$y(t) = \sqrt{\dfrac{1}{1}} = 1.$

■

1.7 Bifurcations

Bifurcations occur when the form of solution changes. We will focus on autonomous first order ODEs, $\dfrac{dy}{dt} = f(y)$.

■ **Example 1.31** An equation like $\dfrac{dy}{dt} = y + k$ has no bifurcations. Regardless of the value of k, there is always one equilibrium value. So, the form of solution doesn't change with k. ■

■ **Example 1.32** An equation like $\dfrac{dy}{dt} = y^2 + 2y + k$ does have a bifurcation. The value of k can change the number of equilibrium solutions.

We can evaluate the general equation $y^2 + 2y + k = 0$, and solve for equilibrium values in terms of k using the quadratic formula.

$y = \dfrac{-2 \pm \sqrt{4 - 4*1*k}}{2*1} = \dfrac{-2 \pm \sqrt{4-4k}}{2}$
There are three cases for solutions based on the value inside the square root:
1) Two real values when $4 - 4k > 0$ -> This results in two equilibrium values
2) One real value when $4 - 4k = 0$ -> This results in one equilibrium value
3) Two complex values when $4 - 4k < 0$ -> This results in no (real) equilibrium values
The bifurcation occurs at case 2), when $k = 1$.

[include phase lines] When we draw a phase line for each case, you can clearly see how the bifurcation causes a completely different visual for the solutions. ■

2. Systems of Differential Equations

We introduced the format used for systems of differential equations in the last chapter. In this chapter, we will learn how to solve them.

2.1 Modeling with Systems

In the last chapter we outlined some of the main models that use systems of differential equations: Predator-Prey and Competing Species. We will start by analyzing these forms. Then, we will expand to second order differential equations - which we can model as a system.

2.1.1 Predator-Prey

Recall: Predator-Prey equations are of the form

$\dfrac{dW}{dt} = -aW + bKW$

$\dfrac{dK}{dt} = cK - dKW$ (W represented Whales, K represented Krill)

We discussed the terms and their meaning in 1.1, here we can use concepts from the last few sections to analyze the long-term behaviors. Specifically, we can locate equilibrium points where the two populations are in equilibrium (no longer changing). Just like the individual DEs, the system is at equilibrium when all derivatives equal zero.

So, we solve $-aW + bKW = 0$ and $cK - dKW = 0$ for values of W and K.

There are two equilibrium solutions, W and $K = 0$, or $K = \dfrac{a}{b}$ and $W = \dfrac{c}{d}$.

Physically speaking, the first of these solutions is nonphysical. Why would we model a population with no Whales or Krill? So, we call this the “trivial solution." It produces no meaningful results, but it is a solution.

We can also introduce plots in the phase plane into our analysis. The phase plane looks at the values of $W(t)$ and $K(t)$ with respect to each other. We can plot the output values of the two functions in the phase plane, but the more useful plot is a direction field. We plot in the slopes from $\dfrac{dW}{dt}$ and $\dfrac{dK}{dt}$. In this example, we might select $W(t)$ as the y-axis and $K(t)$ as the x-axis (since the

Whales are dependent on the Krill as a source of food.) Then, to build a direction field in the phase plane we define $\frac{dW}{dK} = \frac{dW/dt}{dK/dt}$ at each value (K,W).

■ **Example 2.1** For the Predator-Prey system $\frac{dW}{dt} = -.2W + 3KW$ and $\frac{dK}{dt} = 2K - 8KW$, determine any equilibrium solutions, then sketch the direction field in the phase plane, and finally sketch solutions in the phase plane and regular ((t,K) and (t,W) - rough sketches based on the direction field, not solutions) for initial values $(K,W) = (.1,.2)$, $(.5,.03)$, and $(.2,.12)$.

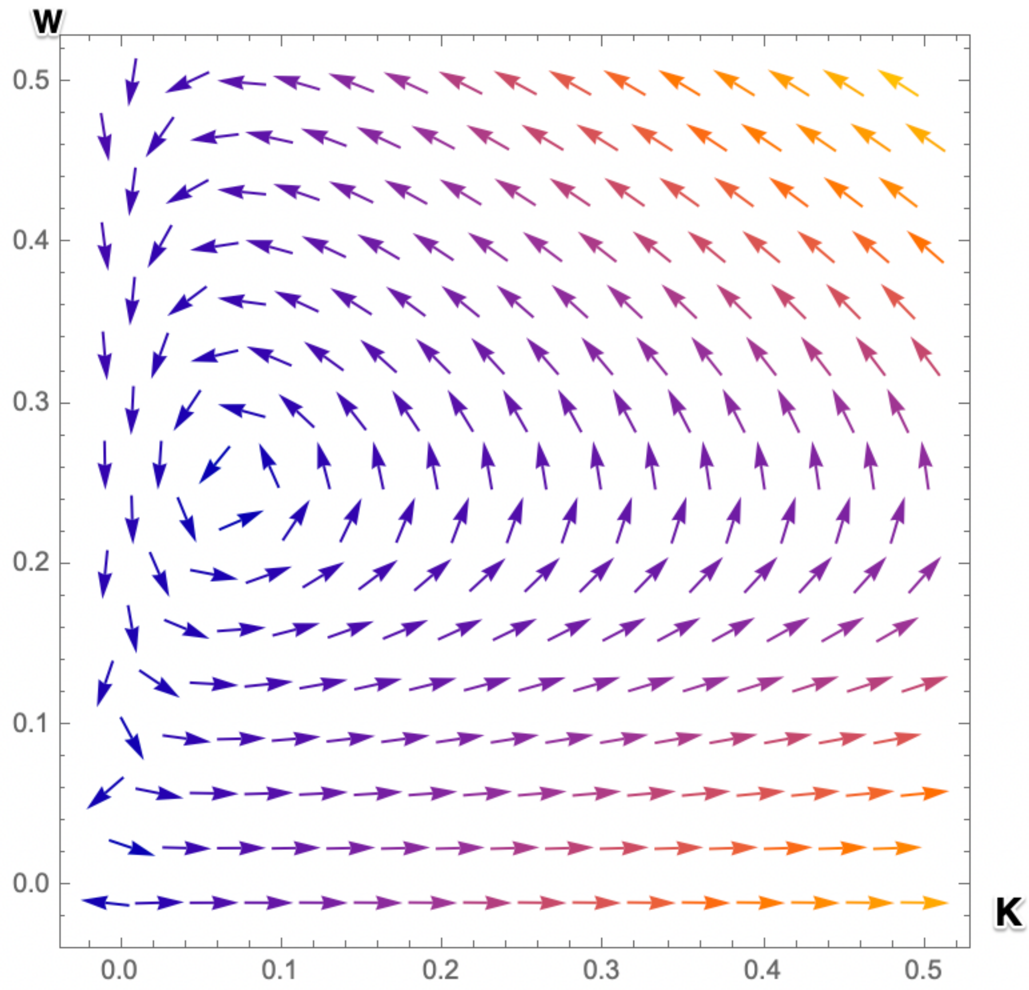
Equilibrium solutions:

$-0.2W + 3KW = 0$, $K = \frac{1}{15}$

$2K - 8KW = 0$, $W = \frac{1}{4}$

There is also the trivial solution, $K = 0$ and $W = 0$.

The direction field sketch uses values of K and W to determine K' and W' at each point, then we calculate W'/K' to determine the slope we sketch. Using software yields a nice visual.



[insert the sketched in solution curves] [insert the sketched solutions]

■

The Predator-Prey problems typically have this type of direction field: where the solutions orbit the equilibrium point. We can modify the Predator-Prey model, as in the text, and observe changes to the behavior.

2.1.2 Competing Species

Recall: Competing Species equations are of the form

$B' = a_1 B(1 - \gamma_1 B - \delta_1 H)$

$H' = a_2 H(1 - \gamma_2 H - \delta_2 B)$ (H represents Humpback whales and B represents Blue whales, both competing for the same food source, Krill - which is not in the model.)

The equilibrium points for the competing species model have more options.

$B' = 0$ when $B = 0$, or $1 - \gamma_1 B = \delta_1 H$.

$H' = 0$ when $H = 0$, or $1 - \gamma_2 H = \delta_2 B$.

The scenarios with B, H, or both $= 0$ are trivial. They are still important equilibrium points, but they do not represent an equilibrium with both species surviving.

So, we seek to find values of B and H that are nonzero. If $H = (1 - \gamma_1 B)/\delta_1$, then $1 - \gamma_2(1 - \gamma_1 B)/\delta_1 = \delta_2 B$.

$1 - \gamma_2/\delta_1 + \gamma_2\gamma_1 B/\delta_1 = \delta_2 B$

$1 - \gamma_2/\delta_1 = B(\delta_2 - \gamma_2\gamma_1/\delta 1)$

$$\frac{1 - \gamma_2/\delta_1}{\delta_2 - \gamma_2\gamma_1/\delta_1} = B$$

$$B = \frac{\delta_1 - \gamma_2}{\delta_1\delta_2 - \gamma_1\gamma_2}$$

$$H = \frac{1 - \gamma_1 \dfrac{\delta_1 - \gamma_2}{\delta_1\delta_2 - \gamma_1\gamma_2}}{\delta_1}$$

We can then plot the direction field in the same manner as the Predator-Prey system.

■ **Example 2.2** Analyzing the Competing Species model $\dfrac{dH}{dt} = 0.2H(1 - 0.5H - 0.25B)$, $\dfrac{dB}{dt} = 0.3B(1 - 0.4B - 0.3H)$, determine any equilibrium solutions, then sketch the direction field in the phase plane, and finally sketch solutions in the phase plane and regular ((t,H) and (t,B) - rough sketches based on the direction field, not solutions) for initial values $(H,B) = (1,.2)$, $(.5,.03)$, and $(.2,.12)$.

Equilibrium solutions exist when $\dfrac{dH}{dt}$ and $\dfrac{dB}{dt} = 0$.

$0.2H(1 - 0.5H - 0.25B) = 0$ when $H = 0$ or $H = 2 - 0.5B$

$0.3B(1 - 0.4B - 0.3H) = 0$ when $B = 0$ or $B = 2.5 - 0.75H$

So, there are equilibrium points at $(0,0)$, $(0,2.5)$, $(2,0)$ (trivial solution, and two nonphysical solutions).

The fourth point is defined by the equations $H = 2 - 0.5B$ and $B = 2.5 - 0.75H$ (the physical equilibrium point).

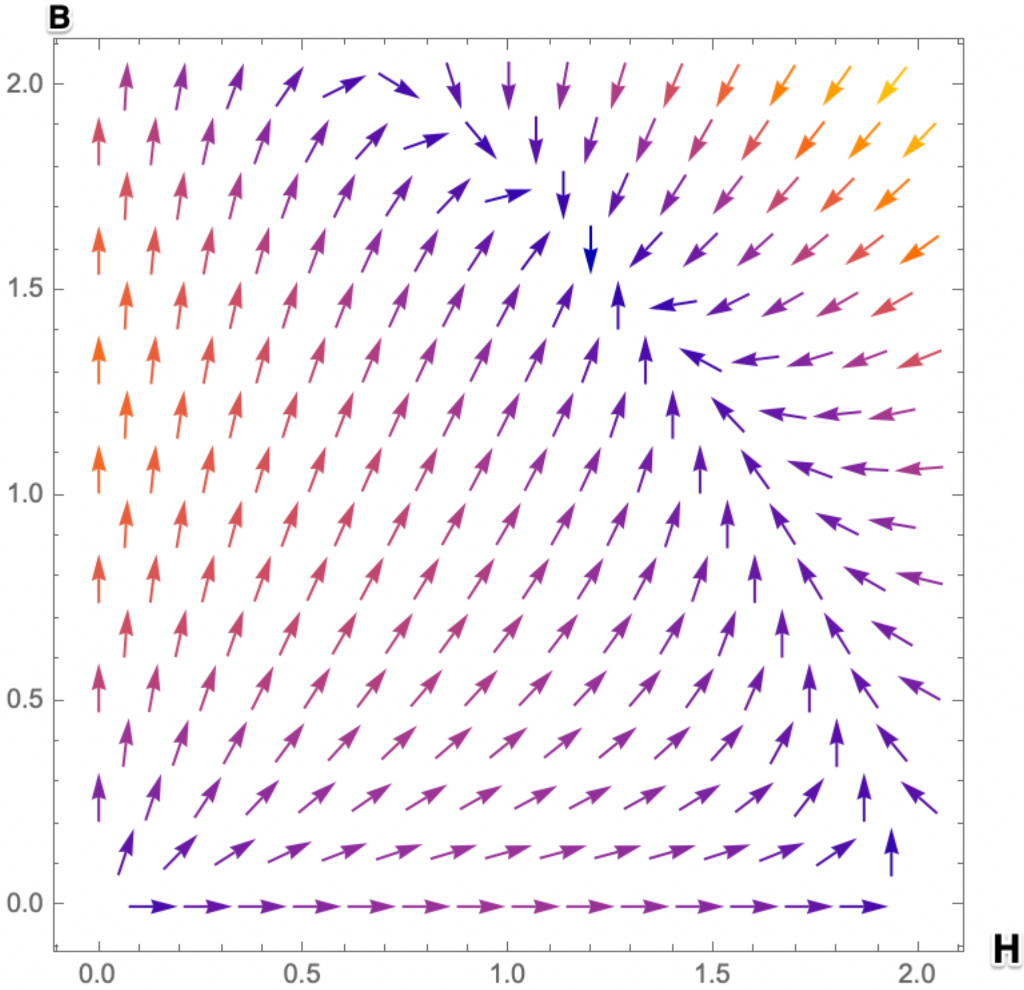
$B = 2.5 - 0.75 * (-0.5B + 2) = 2.5 + 0.375B - 1.5 = 1 + 0.375B$

$0.625B = 1$, $B = 1.6$

$H = -0.75B + 2.5 = -0.75 * 1.6 + 2.5 = 1.3$

Equilibrium point at $(1.3, 1.6)$.

We can sketch the direction field by selecting values of H and B, then plotting B'/H' as in the last example. The visual of the direction field is clearest through a computer, so we'll view that here.



[insert the sketched in solution curves] [insert the sketched solutions]

■

2.1.3 Second Order Differential Equations

Second order differential equations are generally of the form $ay'' + by' + cy = d$, where a, b, c, and d can be constants or functions of t. Rather than solve these outright (which we'll do later), we can manipulate the equation to turn it into a system of two first order differential equations. We do this by defining the derivative, y' as a new variable, v. Then we can define $y'' = f(t, y, v)$ and write the differential equations accordingly.

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = \alpha v + \beta y + \gamma(t)$$

Once we rewrite the equation as a system, we then evaluate it in the manner as the previous problems. For now, we will focus on the oscillator example (which uses x instead of y).

$$x'' + \frac{\gamma}{m}x' + \frac{k}{m}x = 0$$

In this equation, the first order system is written as

$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = \frac{-\gamma}{m}v - \frac{k}{m}x$$

We solve for equilibrium points: $v = 0$, $x = 0$ (when the oscillator is at rest!) This is consistent with the concept of an equlibrium point when $x = 0$ (displacement is zero), and $v = 0$ means it isn't moving, either. This makes sense because the mass should not be moving if it is at equilibrium.

■ **Example 2.3** Analyze the behavior of a mass-spring systems with mass of $4kg$, spring constant of $9N/m$.

1) Without damping

2) With a damping constant of $1kg/s$

3) With a damping constant of $12kg/s$

4) With a damping constant of $20kg/s$

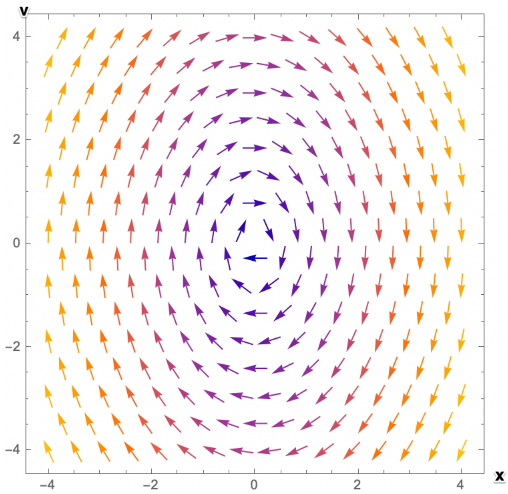
We will construct the equations for each of these separately, and evaluate them.

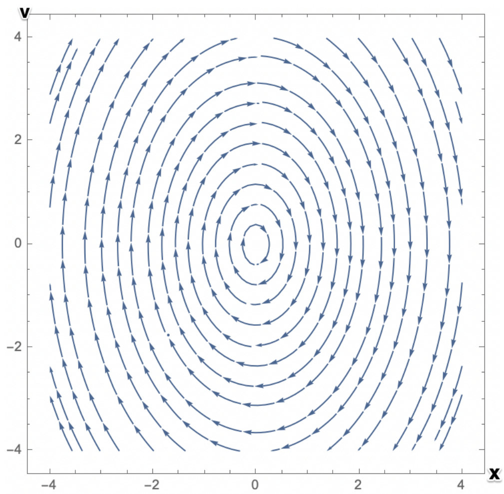
1) Without damping, so $\gamma = 0kg/s$, and the system of equations becomes

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\frac{9}{4}x$$

The only equilibrium point occurs when x and $v = 0$. (As discussed earlier.)





[add solution curves] [add sketch of solutions]

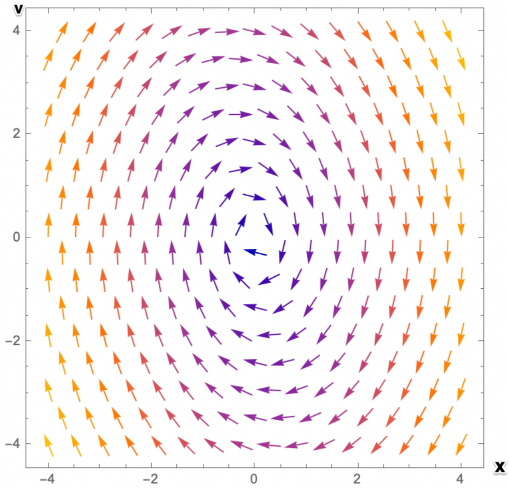
These plots show concentric circles around $(0,0)$. This is an undamped spring-mass system and the equlibrium point is labeled as a stable center.

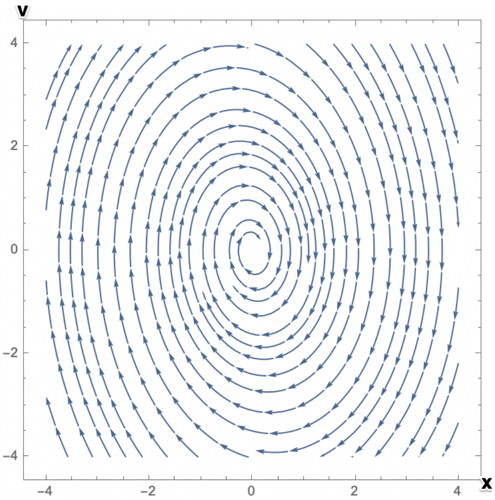
2) With a damping constant of $1kg/s$, the system of equations becomes

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \frac{-1}{4}v - \frac{9}{4}x$$

The only equilibrium point is still at $(0,0)$.





[add solution curves] [add sketch of solutions]

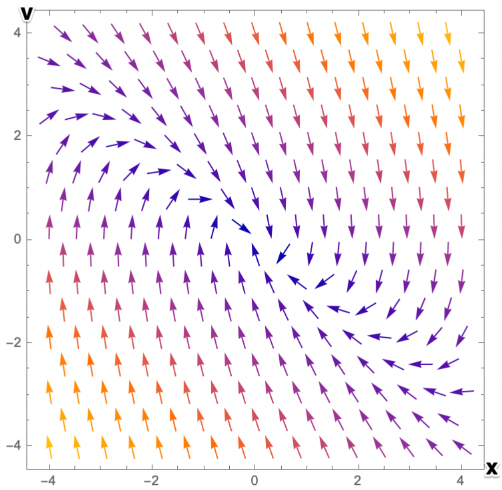
These plots show that the solutions spiral into $(0,0)$. This is an underdamped spring-mass system and the equilibrium point is labeled as an asymptotically stable sink.

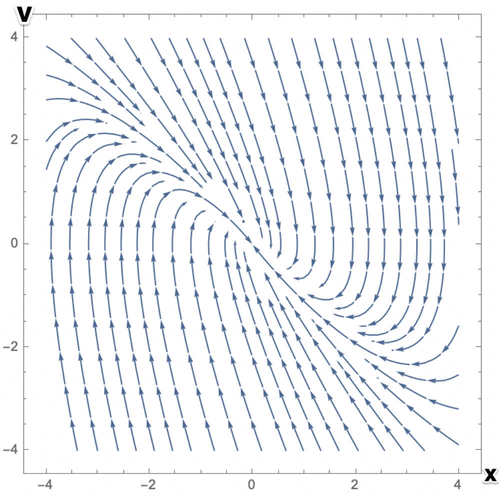
3) With a damping constant of $3kg/s$, the system of equations becomes

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -3v - \frac{9}{4}x$$

The only equilibrium point is still at $(0,0)$.





[add solution curves] [add sketch of solutions]

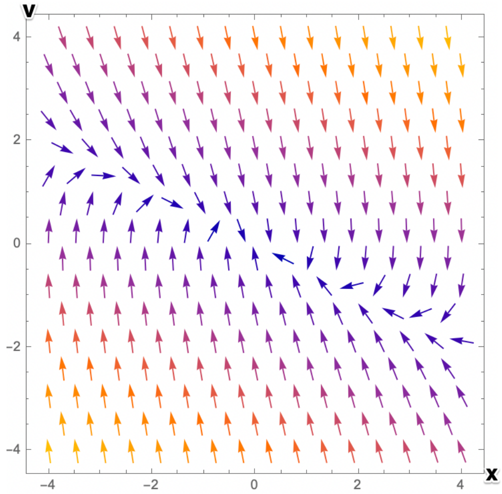
These plots show the solution swirls into $(0,0)$, but it forms an axis that each curve is swirling into. This is a critically damped spring-mass system and the equilibrium point is also a stable node (does not make a full oscillation).

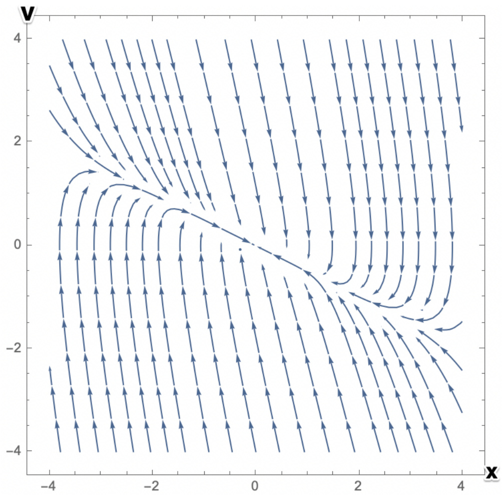
4) With a damping constant of $7kg/s$, the system of equations becomes

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -5v - \frac{9}{4}x$$

The only equilibrium point is still at $(0,0)$.





[add solution curves] [add sketch of solutions]

These plots show the solution curves into $(0,0)$ and there is a clear axis along which the curves approach $(0,0)$. This is an overdamped spring-mass system and the equilibrium point is also a stable node (no oscillation). ■

2.2 The Geometry of Systems

Expanding on the work we did in the last section: direction fields and phase portraits for autonomous systems and nullclines.

2.2.1 Phase Plane Analysis

Building on the work in 2.1, we can construct phase portraits and classify the behavior of equilibrium solutions more generally. We will focus on autonomous systems of differential equations of the form:

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

A **nullcline** is a curve where one of the derivatives in your system is equal to zero.

The x-nullclines are when $\frac{dx}{dt} = f(x,y) = 0$, and the y-nullclines are when $\frac{dy}{dt} = g(x,y) = 0$.

The nullclines are quick curves to sketch in by hand when sketching a phase portrait.

We will use the last example from 2.1 to show this (replacing v with y).

■ **Example 2.4** For the system of differential equations

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -5y - \frac{9}{4}x$$

Determine the nullclines, classify the equilibrium point, and sketch the phase portrait.

Nullclines:

The x-nullcline is $y = 0$.

The y-nullcline is $-5y - 9/4x = 0$, or $y = -9/20x$

Notice that the intersection of these two lines is the equilibrium point $(0,0)$.

We can determine the behavior by evaluating the derivatives along these lines.

First, along the x-nullcline, $y = 0$. When $y > 0$, $\frac{dx}{dt} > 0$, and when $y < 0$ $\frac{dx}{dt} < 0$. So, above this nullcline, the x-component of a direction field points right, and below this nullcline, the x-component of a direction field points left.

Along the y-nullcline, we have a line $y = -9/20x$, or $y + 9/20x = 0$. For values where $y + 9/20x > 0$ (which causes $g(x,y) = -5y - 9/4x < 0$), $\frac{dy}{dt} < 0$, and for values where $y + 9/20x < 0$ (which causes $g(x,y) = -5y - 9/4x > 0$), $\frac{dy}{dt} > 0$. So, values above this nullcline, the y-component of the direction field points down, and values below this nullcline the y-component points up.

From this, we get a rough sketch of our direction field. Which, is consistent with the plots from Mathematica in the previous section. [include sketched direction field with nullclines]

The equilibrium point is a stable node, all solutions curve into it. ■

■ **Example 2.5** For the system of differential equations

$$\frac{dx}{dt} = x^2 - x$$

$$\frac{dy}{dt} = 2y + x$$

Determine the nullclines, classify the equilibrium point, and sketch the phase portrait.

x-nullclines: $x^2 - x = 0$, when $x = 0$ and $x = 1$.

y-nullclines: $2y + x = 0$, or $y = -x/2$.

There are two intersections between these three nullclines. $(0,0)$, and $(1,-1/2)$. There are two equilibrium points for this system.

We can determine the behavior of the direction field by evaluating values around these nullclines.

For the first two nullclines, the x values are fixed. So, we can determine the sign of $\frac{dx}{dt}$ in the three regions: 1) left of $x = 0$, 2) between $x = 0$ and $x = 1$, and 3) right of $x = 1$.

1) Left of $x = 0$, $x^2 - x > 0$. So, the x-component of the direction field vectors will be pointing to the right.

2) Between $x = 0$ and $x = 1$, $x^2 - x < 0$. So, the x-component of the direction field vectors will be pointing to the left.

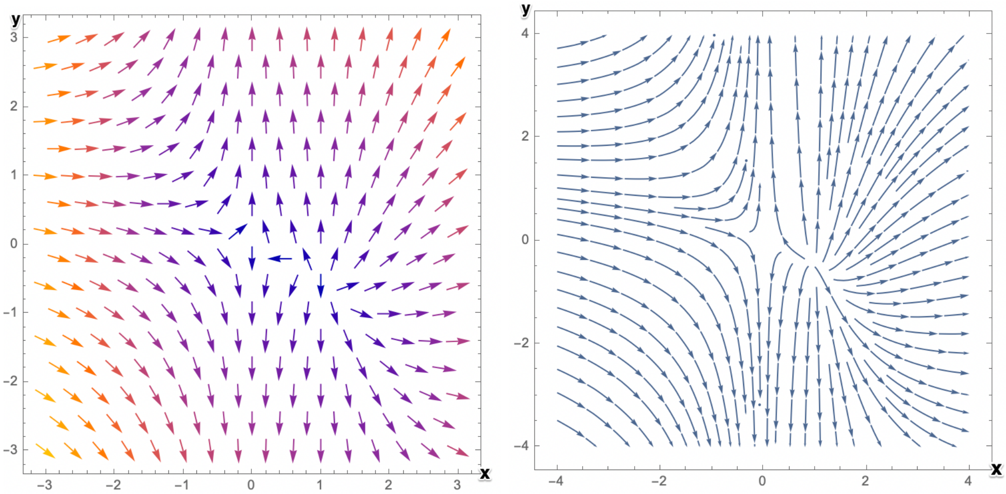
3) Right of $x = 1$, $x^2 - x > 0$. So, the x-component of the direction field vectors will be pointing to the right.

For the third nullcline, $y = -x/2$, or $2y + x = 0$, we want to focus on the term as it was given. Above $y = -x/2$ means that y is larger than $-x/2$, so $2y + x > 0$, and the y-component of the direction field vectors will point up.

So, below $y = -x/2$, $2y + x < 0$, and the y-component of the direction field vectors will point down.

Putting these together, we see there is an unstable saddle at $(0,0)$, and an unstable node at $(1,-1/2)$. [include sketch of direction field with nullclines]

For clarity, here are the plots from Mathematica.



■

Note: this is a nonlinear example. For all our linear systems of differential equations, there will only be one equilibrium value (yay!). We'll focus on nonlinear systems after we finish solving linear ones.

Linear Algebra in a Nutshell

This section is basically a crash-course in matrix algebra. Anyone who has completed a course involving matrix algebra is likely already solid on all these concepts. For anyone who has no idea what matrix algebra is: this section is for you!

Matrix Notation

When working with any kind of system, matrix notation is a handy way to represent the system in a concise manner.

■ **Example 2.6** In algebra, the linear system of equations:

$2x+3y=4$

$3x-7y=2$

Can be rewritten as a linear system with matrices:

$\begin{bmatrix} 2 & 3 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

The first matrix is a matrix of the coefficients in the equations, you may notice that the first column corresponds to the coefficients of x, and the second column corresponds to the coefficients of y. Also, the first row corresponds to the first equation, and the second row corresponds to the second equation. This format is consistent when we write systems of differential equations in a matrix notation. ■

Solving linear systems of equations algebraically using Elimination or Substitution (there are more ways, but for our course we will focus only on these two).

Elimination: eliminate a variable by adding (or subtracting) the two equations.

■ **Example 2.7** Solve the previous linear system using elimination.

$2x+3y=4$

$3x-7y=2$

I would look for a constant to use to eliminate either x or y. I could multiply the first equation by $3/2$ (which would turn the $2x$ into a $3x$ which would allow me to subtract the second equation from the first and elminate x), or by $7/3$ (which would turn the $3y$ into a $7y$ which would allow me to add the two equations and eliminate y).

The first option results in
$3x + 9y/2 = 6$
$3x - 7y = 2$
Subtract the second from the first
$0x + 23y/2 = 4$
Solve for $y = 8/23$.
Substitute y into one of the original equations $2x + 3 * 8/23 = 4$, and solve for $x = 34/23$.
The second option results in
$14x/3 + 7y = 28/3$
$3x - 7y = 2$
Adding these together
$23x/3 + 0y = 34/3$
Solving for $x = 34/23$.
Substituting this back into either equation, $3 * 34/23 - 7y = 2$ and solving for $y = 8/23$. ■

Substitution: Solve for one variable as a function of the other, substitute it into the other equation, and solve for the other variable.

■ **Example 2.8** Solve the problem using substitution. Substitution starts by isolating one variable.
$2x + 3y = 4$
$3x - 7y = 2$
We can take the first equation and solve for $y = \dfrac{4-2x}{3}$
Then, substitute that expression into the second equation in place of y.
$3x - 7\dfrac{(4-2x)}{3} = 2$
Now, solve for x
$3x - 28/3 + 14x/3 = 2$, so $23x/3 = 34/3$, and $x = 34/23$ (regardless of method, we'll always get the same answer.)
Then you can find y the same way we did in the previous example.
■

Using matrix notation for systems of differential equations:

■ **Example 2.9** The system of differential equations:
$\dfrac{dx}{dt} = y$
$\dfrac{dy}{dt} = 3x - 5y$
Can be written using matrices as
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
Notice that the coefficient is zero when we have no x's in the equation. Again, the first row corresponds to the first equation, and the second row corresponds to the second equation. ■

Hopefully you see the pattern in how these matrices multiply, but we'll outline it in the next section.

Elements of a matrix: In a matrix, the elements are denoted by their position (row, column). So, the element in position $(2,3)$ would be in the second row, third column of the matrix.

Matrix Operations

Addition and subtraction: term-by-term.

■ **Example 2.10** $\begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 1+(-4) & 5+1 \\ -3+(-7) & 2+4 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -10 & 6 \end{bmatrix}$ ■

Multiplication: reads right-to-left, unlike most things. Columns (on right) to rows (on left), added together.

■ **Example 2.11** A matrix multiplied by a vector:

$$\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1*1+4*(-5) \\ -1*(-3)+4*2 \end{bmatrix} = \begin{bmatrix} -21 \\ 11 \end{bmatrix}$$ ■

The result of matrix multiplication should have the same number of rows as the first matrix, and the same number of columns as the second matrix. In the last example we had a (2×2) matrix multiplied by a vector that was (2×1). (The notation $(a\times b)$ refers to (# of rows $\times$ # of columns.) The interior dimensions **must** match. In this case $(2\times 2)(2\times 1)$ results in a (2×1). The interior dimensions (columns of the first and rows of the second) were both 2, so the multiplication was possible, and those terms disappear in the resulting matrix.

■ **Example 2.12** Two matrices of differing sizes:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & 3 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} -2*1+4*2+(-5)*3 & 1*1+3*2+(-1)*3 \\ -2*4+4*5+(-5)*6 & 1*4+3*5+(-1)*6 \\ -2*(-3)+4*2+(-5)*(-1) & 1*(-3)+3*2+(-1)*(-1) \end{bmatrix} =$$

$$\begin{bmatrix} -9 & 4 \\ -18 & 13 \\ 20 & 4 \end{bmatrix}$$

Notice: $(3\times 3)(3\times 2) = (3\times 2)$. ■

Exponentials: Break it into multiplication, and complete it manually. $A^2 = A*A$, $A^3 = A*A*A$, etc. Remember that we multiply matrices right-to-left.

■ **Example 2.13** Determine A^2 for $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2*2+3*4 & 4*2+5*4 \\ 2*3+3*5 & 4*3+5*5 \end{bmatrix} = \begin{bmatrix} 16 & 28 \\ 21 & 37 \end{bmatrix}$$ ■

Transpose: Flip the rows and columns.

■ **Example 2.14** Determine the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$ ■

Determinants and Invertible Matrices

The inverse of a matrix, A^{-1}, only exists when the determinant is not equal to zero. The determinant measures how close our matrix is to singular. Attempting to invert singular matrices is the equivalent to dividing by zero in the real numbers. Calculating determinants is actually something you did in Calculus III. Remember: Cross product? Curl? Those were determinant calculations.

The determinant of a (2×2) matrix is the foundation for the remaining determinants (and most of the determinants we calculate will be (2×2).)

The determinant of a (2×2) matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(A) = ad - bc$

Cross product and Curl used a determinant calculation for a (3×3) matrix.

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\det(B) = a(ei - fh) - b(di - fg) + c(dh - eg)$

A matrix is singular if its determinant is equal to zero.

■ **Example 2.15** Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and determine if it is invertible.

$\det(A) = 1(5*9 - 6*8) - 2(4*9 - 6*7) + 3(4*8 - 5*7) = (45 - 48) - 2(36 - 42) + 3(32 - 35) = -3 - 2(-6) + 3(-3) = -3 + 12 - 9 = 0.$

This matrix is not invertible because it is singular! ■

Another important feature of these concepts is linear independence. If the determinant is zero, it means that the columns are not linearly independent or the rows are not linearly independent. In our last example, it isn't obvious, but those rows are not linearly independent of each other. (Row 3 is equal to 2*Row 2 - Row 1.)

We can determine if two vectors are linearly independent using a matrix composed of the vectors and then calculating its determinant.

■ **Example 2.16** Determine if the vectors $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are linearly independent.

$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

$\det(A) = 1*6 - 3(-2) = 6 + 6 = 12 \neq 0$

These vectors are linearly independent because the determinant of the matrix formed by these vectors is not zero. ■

Additionally, when the number of vectors matches the size of the vector (in this case 2), and those vectors are linearly independent, they form a basis of $\mathcal{R}^n$. In this example, the two vectors are linearly independent, and of length 2, so they form a basis of $\mathcal{R}^2$. A standard basis is like $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ in Calculus III, a single 1 along one dimension, and zeroes elsewhere. $\hat{\imath} = \langle 1,0,0 \rangle$, $\hat{\jmath} = \langle 0,1,0 \rangle$, and $\hat{k} = \langle 0,0,1 \rangle$. The vectors in the previous example form a basis, but are not standard basis vectors.

Eigenvalues and Eigenvectors

This is the most important bit for the work we're about to do!

Eigenvalues are a measure of a matrix. Specifically, the eigenvalues of a matrix are special scalar values such that $A\vec{v} = \lambda\vec{v}$. As these values are not dependent on $\vec{v}$, but only on A, we must algebraically manipulate the expression and solve for λ.

We solve the equation $A\vec{v} - \lambda\vec{v} = \vec{0}$. This requires the introduction of the identity matrix: an identity matrix, I, contains 1's on the diagonal, and zeroes elsewhere. This allows us to solve $(A - \lambda I)\vec{v} = \vec{0}$.

Assuming that A is nonsingular, we seek values of λ that cause $A - \lambda I$ to become singular.

We then solve for the determinant of $A - \lambda I$, set it equal to zero, and solve for λ. The polynomial of λ which results from this calculation is called the characteristic polynomial.

When the matrix is singular, there is a dependency between the columns to cause the $\vec{0}$ on the right side of the equation. Once we solve for λ, there will be an associated eigenvector for each value of λ that satisfies the equation.

■ **Example 2.17** Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -\lambda & 2 \\ 8 & -\lambda \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - 2*8 = \lambda^2 - 16$
When does $\det(A - \lambda I) = 0$?
$\lambda_1 = -4$
$\lambda_2 = 4$
Solve for the eigenvectors associated with each eigenvalue.
1) $\lambda_1 = -4$, so $A - \lambda I = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$

This is a singular matrix because column 1 is 2*column 2. So the eigenvector needed would ensure that $4x + 2y = 0$, or $y = -2x$.

An example vector that has that relationship is $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So, the eigenvalue $\lambda_1 = -4$ has an associated eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

2) $\lambda_2 = 4$, so $A - \lambda I = \begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix}$

This is a singular matrix because column 1 is -2*column 2. So, the eigenvector needed would ensure that $-4x + 2y = 0$, or $y = 2x$.

An example vector that has that relationship is $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So the eigenvalue $\lambda_2 = 4$ has an associated eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

■

■ **Example 2.18** Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$
$\det(A - \lambda I) = (1-\lambda)(4-\lambda) - 2*3 = 4 - \lambda - 4\lambda + \lambda^2 - 6 = \lambda^2 - 5\lambda - 2$
When does $\det(A - \lambda I) = 0$? Use the quadratic formula!
$$\frac{5 \pm \sqrt{25 - 4*1*(-2)}}{2*1} = \frac{5 \pm \sqrt{33}}{2}$$
$$\lambda_1 = \frac{5 - \sqrt{33}}{2}$$
$$\lambda_2 = \frac{5 + \sqrt{33}}{2}$$
Solve for the eigenvectors associated with each eigenvalue.

1) $\lambda_1 = \dfrac{5-\sqrt{33}}{2}$, so $A - \lambda I = \begin{bmatrix} 1 - \dfrac{5-\sqrt{33}}{2} & 2 \\ 3 & 4 - \dfrac{5-\sqrt{33}}{2} \end{bmatrix}$

We know this is a singular matrix, but it is less obviously so. We can select either row to construct the relationship between x and y in the associated eigenvector.
$$\left(1 - \frac{5-\sqrt{33}}{2}\right)x + 2y = 0, \text{ or } \frac{\sqrt{33}-3}{2}x + 2y = 0, \text{ or } y = \frac{3-\sqrt{33}}{4}x$$

Then, an associated eigenvector would be $\vec{v}_1 = \begin{bmatrix} 1 \\ \dfrac{3-\sqrt{33}}{4} \end{bmatrix}$

2) $\lambda_2 = \dfrac{5+\sqrt{33}}{2}$, so $A - \lambda I = \begin{bmatrix} 1 - \dfrac{5+\sqrt{33}}{2} & 2 \\ 3 & 4 - \dfrac{5+\sqrt{33}}{2} \end{bmatrix}$

We again know this is a singular matrix, but we'll have to just use one of the rows to define the eigenvector.

$\left(1-\frac{5+\sqrt{33}}{2}\right)x+2y=0$, or $\frac{-\sqrt{33}-3}{2}x+2y=0$, or $y=\frac{3+\sqrt{33}}{4}$

Then, an associated eigenvector would be $\vec{v}_2=\begin{bmatrix}1\\ \frac{3+\sqrt{33}}{4}\end{bmatrix}$

The eigenvalues and their associated eigenvectors are then $\lambda_1=\frac{5-\sqrt{33}}{2}$, $\vec{v}_1=\begin{bmatrix}1\\ \frac{3-\sqrt{33}}{4}\end{bmatrix}$, and $\lambda_2=\frac{5+\sqrt{33}}{2}$, $\vec{v}_2=\begin{bmatrix}1\\ \frac{3+\sqrt{33}}{4}\end{bmatrix}$.

■

2.3 Numerical Methods for Solving Systems

When solving a system of differential equations, it's important to feel comfortable with the matrix notation - we'll need to use it to apply numerical methods to a system of equations.

We will only use Euler's method in this course, because the other techniques are far more complicated. While those techniques are more valuable in application, Euler's should give you the idea so you can explore other techniques more easily in the future.

The method is the same, but what changes is the format.

$\vec{y}_{n+1}=\vec{y}_n+h\vec{f}(t_n,\vec{y}_n)$

$t_{n+1}=t_n+h$ (as before)

Notice that the equation looks the same, but there are vectors where there were previously scalars. That's the big change. Which is why it is important to be comfortable using matrices.

The good news is that we'll focus on simpler problems: autonomous linear systems.

$\begin{bmatrix}x'\\ y'\end{bmatrix}=A\begin{bmatrix}x\\ y\end{bmatrix}$

The work we do will generalize to systems that are nonlinear and for systems that are nonautonomous. However, you can enjoy that in another course, or for fun (on your own time).

■ **Example 2.19** Use Euler's method for systems to approximate the values of x and y at $t=0.6$ using $h=0.2$ for the system

$\frac{dx}{dt}=-2x+4y$

$\frac{dy}{dt}=3x+7y$

with initial value $x(0.2)=1$, $y(0.2)=4$.

First, it's helpful to construct these into a matrix notation.

$\vec{y}'=\begin{bmatrix}-2 & 4\\ 3 & 7\end{bmatrix}\vec{y}$ with $vecy(0.2)=\begin{bmatrix}1\\ 4\end{bmatrix}$

Thus, $\vec{y}_0=\begin{bmatrix}1\\ 4\end{bmatrix}$, and $\vec{f}(\vec{y})=\begin{bmatrix}-2 & 4\\ 3 & 7\end{bmatrix}\vec{y}$. (Note, generally speaking $\vec{f}$ is also dependent on t, but this is an autonomous system - no t dependence in $\vec{f}$.)

Applying Euler's method, $\vec{y}_1=\vec{y}_0+h*\vec{f}(\vec{y}_0)$

$\vec{y}_1=\begin{bmatrix}1\\ 4\end{bmatrix}+0.2*\begin{bmatrix}-2 & 4\\ 3 & 7\end{bmatrix}\begin{bmatrix}1\\ 4\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}+0.2*\begin{bmatrix}1*(-2)+4*4\\ 1*3+4*7\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}+0.2*\begin{bmatrix}14\\ 31\end{bmatrix}$

$=\begin{bmatrix}1\\ 4\end{bmatrix}+\begin{bmatrix}2.8\\ 0.8\end{bmatrix}=\begin{bmatrix}3.8\\ 4.8\end{bmatrix}$

So the approximation to $\vec{y}(0.4) \approx \vec{y}_1 = \begin{bmatrix} 3.8 \\ 4.8 \end{bmatrix}$.

One more step to reach $\vec{y}(0.6)$.

$\vec{y}_2 = \vec{y}_1 + h * \vec{f}(\vec{y}_1)$

$\vec{y}_2 = \begin{bmatrix} 3.8 \\ 4.8 \end{bmatrix} + 0.2 * \begin{bmatrix} -2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 3.8 \\ 4.8 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 4.8 \end{bmatrix} + 0.2 \begin{bmatrix} 3.8*(-2)+4.8*4 \\ 3.8*3+4.8*7 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 4.8 \end{bmatrix} + 0.2 \begin{bmatrix} 11.6 \\ 45 \end{bmatrix}$

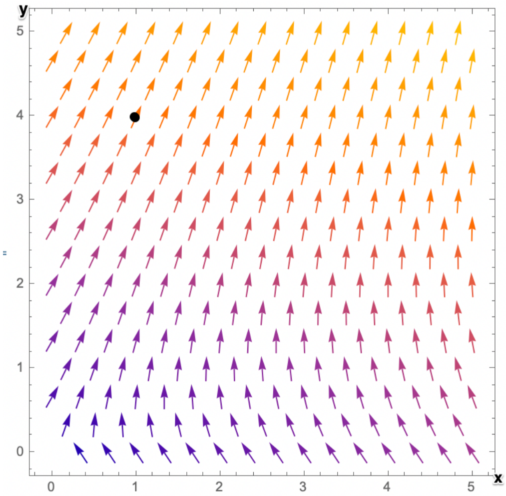
$= \begin{bmatrix} 3.8 \\ 4.8 \end{bmatrix} + \begin{bmatrix} 2.32 \\ 9 \end{bmatrix} = \begin{bmatrix} 6.12 \\ 13.8 \end{bmatrix}$

So, the approximation to $\vec{y}(0.6) \approx \vec{y}_2 = \begin{bmatrix} 6.12 \\ 13.8 \end{bmatrix}$

More steps: continue the process.

We can compare this with the exact solution later (which we will solve for soon!).

For now, we can look at the direction field for confirmation. Here is the direction field plot from Mathematica.



We can see in the direction field that if we start at the point $(1,4)$ in (x,y), both values should increase as we move along the vector field lines. ■

2.4 Solving Systems Analytically

We will solve decoupled and partially coupled systems analytically. In the next chapter, we will expand on these solutions using the matrix notation and linear algebra concepts.

2.4.1 Decoupled Systems

Decoupled systems are simply two differential equations that are not dependent on each other.

$$\frac{dx}{dt} = f(x)$$

$$\frac{dy}{dt} = g(y)$$

Decoupled systems can be solved using the techniques from Chapter 1. Solve each equation independently.

■ **Example 2.20** Solve the system

$$\frac{dx}{dt} = 2x$$

$$\frac{dy}{dt} = 3y - 7$$

We solve each equation separately.

$\frac{dx}{dt} = 2x$ is separable, so we can solve it using a separation of variables.
$\frac{dx}{x} = 2dt$
Integrate both sides: $\ln|x| = 2t + C$
Solve for $x(t) = e^{2t+C} = ke^{2t}$ where $k = e^C$.
$\frac{dy}{dt} = 3y - 7$ can be solved using a separation of variables also, but we will rewrite it and use integrating factor.
$y' - 3y = 7$, $\mu(t) = e^{\int -3dt} = e^{-3t}$
$\frac{d}{dt}\left[e^{-3t}y\right] = 7e^{-3t}$
$e^{-3t}y = \frac{-7}{3}e^{-3t} + C$
$y(t) = \frac{-7}{3} + Ce^{3t}$
Done. No new techniques. ■

2.4.2 Partially Coupled Systems

A partially coupled system only has one equation with dependence on the other.
$\frac{dx}{dt} = f(x)$
$\frac{dy}{dt} = g(x,y)$
or
$\frac{dx}{dt} = f(x,y)$
$\frac{dy}{dt} = g(y)$
are both partially coupled systems. We solve partially coupled systems similarly to the decoupled systems. Solve the equation that is only dependent on itself first, then solve the partially coupled equation.

■ **Example 2.21** Solve the system
$\frac{dx}{dt} = 3 + x$
$\frac{dy}{dt} = x + 2y$
First, we solve $\frac{dx}{dt}$ because it has no dependence on y.
$\frac{dx}{dt} = 3 + x$ is a separable equation, so we solve it using a separation of variables.
$\frac{dx}{3+x} = dt$ integrate both sides
$\ln|3 + x| = t + C$ then solve for $x(t)$
$3 + x = e^{t+C}$, so $x(t) = e^{t+C} - 3$.
Then, substitute $x(t)$ into $\frac{dy}{dt}$ to solve for y.
$\frac{dy}{dt} = e^{t+C} - 3 + y$, we'll need to use integrating factor to solve this equation.
$y' - y = e^{t+C} - 3$, $\mu(t) = e^{\int -dt} = e^{-t}$
$\frac{d}{dt}[e^{-t}y] = e^C - 3e^{-t}$ Integrate both sides with respect to t.
$e^{-t}y = te^C + 3e^{-t} + C_2$
$y(t) = te^{t+C} + 3 + C_2e^t$

Note: The constant C will be defined by the initial value on x, and the constant C_2 will be defined by the initial value on y. ■

3. Linear Systems

This chapter will focus in on solution techniques for linear systems of differential equations. First, the section on linear algebra (which was also included in Chapter 2 without links in the TOC.)

3.1 Linear Algebra in a Nutshell

This section is basically a crash-course in matrix algebra. Anyone who has completed a course involving matrix algebra is likely already solid on all these concepts. For anyone who has no idea what matrix algebra is: this section is for you!

3.1.1 Matrix Notation

When working with any kind of system, matrix notation is a handy way to represent the system in a concise manner.

■ **Example 3.1** In algebra, the linear system of equations:

$2x+3y=4$

$3x-7y=2$

Can be rewritten as a linear system with matrices:

$\begin{bmatrix} 2 & 3 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

The first matrix is a matrix of the coefficients in the equations, you may notice that the first column corresponds to the coefficients of x, and the second column corresponds to the coefficients of y. Also, the first row corresponds to the first equation, and the second row corresponds to the second equation. This format is consistent when we write systems of differential equations in a matrix notation. ■

Solving linear systems of equations algebraically using Elimination or Substitution (there are more ways, but for our course we will focus only on these two).

Elimination: eliminate a variable by adding (or subtracting) the two equations.

■ **Example 3.2** Solve the previous linear system using elimination.

$2x+3y=4$
$3x-7y=2$

I would look for a constant to use to eliminate either x or y. I could multiply the first equation by $3/2$ (which would turn the $2x$ into a $3x$ which would allow me to subtract the second equation from the first and elminate x), or by $7/3$ (which would turn the $3y$ into a $7y$ which would allow me to add the two equations and eliminate y).

The first option results in
$3x+9y/2=6$
$3x-7y=2$
Subtract the second from the first
$0x+23y/2=4$
Solve for $y=8/23$.
Substitute y into one of the original equations $2x+3*8/23=4$, and solve for $x=34/23$.
The second option results in
$14x/3+7y=28/3$
$3x-7y=2$
Adding these together
$23x/3+0y=34/3$
Solving for $x=34/23$.
Substituting this back into either equation, $3*34/23-7y=2$ and solving for $y=8/23$. ■

Substitution: Solve for one variable as a function of the other, substitute it into the other equation, and solve for the other variable.

■ **Example 3.3** Solve the problem using substitution. Substitution starts by isolating one variable.
$2x+3y=4$
$3x-7y=2$
We can take the first equation and solve for $y=\dfrac{4-2x}{3}$
Then, substitute that expression into the second equation in place of y.
$3x-7\dfrac{(4-2x)}{3}=2$
Now, solve for x
$3x-28/3+14x/3=2$, so $23x/3=34/3$, and $x=34/23$ (regardless of method, we'll always get the same answer.)
Then you can find y the same way we did in the previous example.

■

Using matrix notation for systems of differential equations:

■ **Example 3.4** The system of differential equations:
$\dfrac{dx}{dt}=y$
$\dfrac{dy}{dt}=3x-5y$
Can be written using matrices as
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
Notice that the coefficient is zero when we have no x's in the equation. Again, the first row corresponds to the first equation, and the second row corresponds to the second equation. ■

Hopefully you see the pattern in how these matrices multiply, but we'll outline it in the next section.

Elements of a matrix: In a matrix, the elements are denoted by their position (row, column). So, the element in position $(2,3)$ would be in the second row, third column of the matrix.

3.1.2 Matrix Operations

Addition and subtraction: term-by-term.

■ **Example 3.5** $\begin{bmatrix} 1 & 5 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 1+(-4) & 5+1 \\ -3+(-7) & 2+4 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -10 & 6 \end{bmatrix}$ ■

Multiplication: reads right-to-left, unlike most things. Columns (on right) to rows (on left), added together.

■ **Example 3.6** A matrix multiplied by a vector:

$$\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1*1+4*(-5) \\ -1*(-3)+4*2 \end{bmatrix} = \begin{bmatrix} -21 \\ 11 \end{bmatrix}$$ ■

The result of matrix multiplication should have the same number of rows as the first matrix, and the same number of columns as the second matrix. In the last example we had a (2×2) matrix multiplied by a vector that was (2×1). (The notation $(a\times b)$ refers to (# of rows $\times$ # of columns.) The interior dimensions **must** match. In this case $(2\times 2)(2\times 1)$ results in a (2×1). The interior dimensions (columns of the first and rows of the second) were both 2, so the multiplication was possible, and those terms disappear in the resulting matrix.

■ **Example 3.7** Two matrices of differing sizes:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 4 & 3 \\ -5 & -1 \end{bmatrix} = \begin{bmatrix} -2*1+4*2+(-5)*3 & 1*1+3*2+(-1)*3 \\ -2*4+4*5+(-5)*6 & 1*4+3*5+(-1)*6 \\ -2*(-3)+4*2+(-5)*(-1) & 1*(-3)+3*2+(-1)*(-1) \end{bmatrix} =$$
$$\begin{bmatrix} -9 & 4 \\ -18 & 13 \\ 20 & 4 \end{bmatrix}$$

Notice: $(3\times 3)(3\times 2) = (3\times 2)$. ■

Exponentials: Break it into multiplication, and complete it manually. $A^2 = A*A$, $A^3 = A*A*A$, etc. Remember that we multiply matrices right-to-left.

■ **Example 3.8** Determine A^2 for $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2*2+3*4 & 4*2+5*4 \\ 2*3+3*5 & 4*3+5*5 \end{bmatrix} = \begin{bmatrix} 16 & 28 \\ 21 & 37 \end{bmatrix}$$ ■

Transpose: Flip the rows and columns.

■ **Example 3.9** Determine the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$ ■

3.1.3 Determinants and Invertible Matrices

The inverse of a matrix, A^{-1}, only exists when the determinant is not equal to zero. The determinant measures how close our matrix is to singular. Attempting to invert singular matrices is the equivalent to dividing by zero in the real numbers. Calculating determinants is actually something you did in Calculus III. Remember: Cross product? Curl? Those were determinant calculations.

The determinant of a (2×2) matrix is the foundation for the remaining determinants (and most of the determinants we calculate will be (2×2).)

The determinant of a (2×2) matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = ad - bc$

Cross product and Curl used a determinant calculation for a (3×3) matrix.

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\det(B) = a(ei - fh) - b(di - fg) + c(dh - eg)$

A matrix is singular if its determinant is equal to zero.

■ **Example 3.10** Calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and determine if it is invertible.

$\det(A) = 1(5*9 - 6*8) - 2(4*9 - 6*7) + 3(4*8 - 5*7) = (45 - 48) - 2(36 - 42) + 3(32 - 35) = -3 - 2(-6) + 3(-3) = -3 + 12 - 9 = 0.$

This matrix is not invertible because it is singular! ■

Another important feature of these concepts is linear independence. If the determinant is zero, it means that the columns are not linearly independent or the rows are not linearly independent. In our last example, it isn't obvious, but those rows are not linearly independent of each other. (Row 3 is equal to 2*Row 2 - Row 1.)

We can determine if two vectors are linearly independent using a matrix composed of the vectors and then calculating its determinant.

■ **Example 3.11** Determine if the vectors $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are linearly independent.

$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$

$\det(A) = 1*6 - 3(-2) = 6 + 6 = 12 \neq 0$

These vectors are linearly independent because the determinant of the matrix formed by these vectors is not zero. ■

Additionally, when the number of vectors matches the size of the vector (in this case 2), and those vectors are linearly independent, they form a basis of $\mathcal{R}^n$. In this example, the two vectors are linearly independent, and of length 2, so they form a basis of $\mathcal{R}^2$. A standard basis is like $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ in Calculus III, a single 1 along one dimension, and zeroes elsewhere. $\hat{\imath} = \langle 1,0,0 \rangle$, $\hat{\jmath} = \langle 0,1,0 \rangle$, and $\hat{k} = \langle 0,0,1 \rangle$. The vectors in the previous example form a basis, but are not standard basis vectors.

3.1.4 Eigenvalues and Eigenvectors

This is the most important bit for the work we're about to do!

Eigenvalues are a measure of a matrix. Specifically, the eigenvalues of a matrix are special scalar values such that $A\vec{v} = \lambda\vec{v}$. As these values are not dependent on $\vec{v}$, but only on A, we must algebraically manipulate the expression and solve for λ.

We solve the equation $A\vec{v} - \lambda\vec{v} = \vec{0}$. This requires the introduction of the identity matrix: an identity matrix, I, contains 1's on the diagonal, and zeroes elsewhere. This allows us to solve $(A - \lambda I)\vec{v} = \vec{0}$.

Assuming that A is nonsingular, we seek values of λ that cause $A - \lambda I$ to become singular.

We then solve for the determinant of $A - \lambda I$, set it equal to zero, and solve for λ. The polynomial of λ which results from this calculation is called the characteristic polynomial.

When the matrix is singular, there is a dependency between the columns to cause the $\vec{0}$ on the right side of the equation. Once we solve for λ, there will be an associated eigenvector for each value of λ that satisfies the equation.

■ **Example 3.12** Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -\lambda & 2 \\ 8 & -\lambda \end{bmatrix}$
$\det(A - \lambda I) = \lambda^2 - 2 * 8 = \lambda^2 - 16$
When does $\det(A - \lambda I) = 0$?
$\lambda_1 = -4$
$\lambda_2 = 4$
Solve for the eigenvectors associated with each eigenvalue.
1) $\lambda_1 = -4$, so $A - \lambda I = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$

This is a singular matrix because column 1 is 2∗column 2. So the eigenvector needed would ensure that $4x + 2y = 0$, or $y = -2x$.

An example vector that has that relationship is $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

So, the eigenvalue $\lambda_1 = -4$ has an associated eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

2) $\lambda_2 = 4$, so $A - \lambda I = \begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix}$

This is a singular matrix because column 1 is -2∗column 2. So, the eigenvector needed would ensure that $-4x + 2y = 0$, or $y = 2x$.

An example vector that has that relationship is $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

So the eigenvalue $\lambda_2 = 4$ has an associated eigenvector $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

■

■ **Example 3.13** Determine the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$
$\det(A - \lambda I) = (1-\lambda)(4-\lambda) - 2 * 3 = 4 - \lambda - 4\lambda + \lambda^2 - 6 = \lambda^2 - 5\lambda - 2$
When does $\det(A - \lambda I) = 0$? Use the quadratic formula!

$$\frac{5 \pm \sqrt{25 - 4 * 1 * (-2)}}{2 * 1} = \frac{5 \pm \sqrt{33}}{2}$$

$$\lambda_1 = \frac{5 - \sqrt{33}}{2}$$

$$\lambda_2 = \frac{5 + \sqrt{33}}{2}$$

Solve for the eigenvectors associated with each eigenvalue.

1) $\lambda_1 = \dfrac{5 - \sqrt{33}}{2}$, so $A - \lambda I = \begin{bmatrix} 1 - \dfrac{5 - \sqrt{33}}{2} & 2 \\ 3 & 4 - \dfrac{5 - \sqrt{33}}{2} \end{bmatrix}$

We know this is a singular matrix, but it is less obviously so. We can select either row to construct the relationship between x and y in the associated eigenvector.

$$\left(1 - \frac{5 - \sqrt{33}}{2}\right)x + 2y = 0, \text{ or } \frac{\sqrt{33} - 3}{2}x + 2y = 0, \text{ or } y = \frac{3 - \sqrt{33}}{4}x$$

Then, an associated eigenvector would be $\vec{v}_1 = \begin{bmatrix} 1 \\ \dfrac{3-\sqrt{33}}{4} \end{bmatrix}$

2) $\lambda_2 = \dfrac{5+\sqrt{33}}{2}$, so $A - \lambda I = \begin{bmatrix} 1 - \dfrac{5+\sqrt{33}}{2} & 2 \\ 3 & 4 - \dfrac{5+\sqrt{33}}{2} \end{bmatrix}$

We again know this is a singular matrix, but we'll have to just use one of the rows to define the eigenvector.

$\left(1 - \dfrac{5+\sqrt{33}}{2}\right)x + 2y = 0$, or $\dfrac{-\sqrt{33}-3}{2}x + 2y = 0$, or $y = \dfrac{3+\sqrt{33}}{4}$

Then, an associated eigenvector would be $\vec{v}_2 = \begin{bmatrix} 1 \\ \dfrac{3+\sqrt{33}}{4} \end{bmatrix}$

The eigenvalues and their associated eigenvectors are then $\lambda_1 = \dfrac{5-\sqrt{33}}{2}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ \dfrac{3-\sqrt{33}}{4} \end{bmatrix}$, and $\lambda_2 = \dfrac{5+\sqrt{33}}{2}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ \dfrac{3+\sqrt{33}}{4} \end{bmatrix}$.

3.2 Planar Systems

Planar systems are (2×2) systems of first order linear differential equations.

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

or $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ which we write concisely as $\vec{x}' = A\vec{x}$.

For a (2×2) first order linear system, if $\det(A) \neq 0$, the equilibrium solution is at $(0,0)$. If $\det(A) = 0$, there will be a line of equilibrium solutions formed by the relationship between x and y. Recall: a singular matrix has infinite solutions to $A\vec{x} = \vec{0}$ because the rows and/or columns are dependent on each other. So, there will be a relationship between x and y that forms the line of solutions.

■ **Example 3.14** Determine equilibrium values for the system $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Check the determinant: $\det(A) = 1 * 4 - 2 * 2 = 0$. The matrix A is singular, so there are infinite solutions to $A\vec{x} = \vec{0}$. (Row 2 = 2*Row 1, and the same for the columns.)

So, the line formed by solving either equation will be the line of equilibrium solutions. $x + 2y = 0$ or $y = -x/2$.

■

3.2.1 Solutions of Planar Systems

Recall: Eigenvalues. $A\vec{x} = \lambda\vec{x}$.

We're attempting to solve $\vec{x}' = A\vec{x}$ (unknown), but we could solve $x' = kx$ using earlier methods, and we found that this was an exponential relationship $x(t) = Ce^{kt}$.

If we use the eigenvalues of the matrix A, we can construct the solution similarly. $\vec{x}' = A\vec{x} = \lambda\vec{x}$. $\vec{x}$ is a vector function dependent on t, and so the same form of solutions results using our eigenvalues of A.

We construct solutions by solving for the eigenvalues and their associated eigenvectors.

$\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$

Note: This form is for two, distinct, real eigenvalues λ_1 and λ_2. We will expand our solution forms for the other two cases (complex eigenvalues and repeated eigenvalues) in later sections.

■ **Example 3.15** Determine the general solution for the linear system $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

First, solve for the eigenvalues of A (Note: if A is singular, one of your eigenvalues will be zero.)

$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (4-\lambda)(3-\lambda) - 1*2 = 0$

The characteristic polynomial is $\lambda^2 - 7\lambda + 10 = 0$

Solving for $\lambda = \dfrac{7 \pm \sqrt{49 - 4*1*10}}{2*1}$

$\lambda = \dfrac{7 \pm \sqrt{9}}{2}$

$\lambda_1 = 2$, $\lambda_2 = 5$ (two distinct, real eigenvalues.)

Second, solve for the associated eigenvalues.

$\lambda_1 = 2$, substitute it into $A - \lambda I$

$(A - \lambda_1 I)\vec{x} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\vec{x} = \vec{0}$

The two rows are exactly the same, so the relationship is $2x + y = 0$, or $y = -2x$. We represent this in an eigenvector by substituting $x = 1$, causing $y = -2$ to form the eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

The first term of our solution $\vec{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$\lambda_2 = 5$, substitute it into $A - \lambda I$

$(A - \lambda_2 I)\vec{x} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$

These rows are the same relation, but Row 2 = -2*Row 1. We can use either row to form the linear relationship. $-x + y = 0$, or $y = x$. So, the eigenvector is $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then the second term of our solution is $\vec{x}_2(t) = e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Finally, we combine the two solutions to form the general solution for $\vec{x}' = A\vec{x}$, $\vec{x}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ■

■ **Example 3.16** Determine the solution for the linear system $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}$ and $x(0) = 2, y(0) = 4$.

First, solve for the eigenvalues of A.

$A - \lambda I = \begin{bmatrix} -1-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (-1-\lambda)(4-\lambda) - 2*1 = 0$

The characteristic polynomial is $\lambda^2 - 3\lambda - 6 = 0$

Solving for $\lambda = \dfrac{3 \pm \sqrt{9 - 4*1*(-6)}}{2*1}$

$\lambda = \dfrac{3 \pm \sqrt{33}}{2}$

$\lambda_1 = \frac{3-\sqrt{33}}{2}$, $\lambda_2 = \frac{3+\sqrt{33}}{2}$ (two distinct, real eigenvalues. They aren't always pretty, so it's good practice using weird ones.)

Second, solve for the associated eigenvalues.

$\lambda_1 = \frac{3-\sqrt{33}}{2}$, substitute it into $A-\lambda I$

$$(A-\lambda_1 I)\vec{x} = \begin{bmatrix} \frac{-5+\sqrt{33}}{2} & 2 \\ 1 & \frac{5+\sqrt{33}}{2} \end{bmatrix} \vec{x} = \vec{0}$$

The two rows are the same relationship, but it's far less obvious when there are fraction forms. We can use either row, but we'll use the second one this time to keep the algebra a little simpler. The relation in the second row is $x + \frac{5+\sqrt{33}}{2}y = 0$, or $y = \frac{-2}{5+\sqrt{33}}x$. We represent this in an eigenvector by substituting $x = 1$, causing $y = \frac{-2}{5+\sqrt{33}}$ to form the eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{-2}{5+\sqrt{33}} \end{bmatrix}$.

The first term of our solution $\vec{x}_1(t) = e^{(3-\sqrt{33})t/2} \begin{bmatrix} 1 \\ \frac{-2}{5+\sqrt{33}} \end{bmatrix}$.

$\lambda_2 = \frac{3+\sqrt{33}}{2}$, substitute it into $A-\lambda I$

$$(A-\lambda_2 I)\vec{x} = \begin{bmatrix} \frac{-5-\sqrt{33}}{2} & 2 \\ 1 & \frac{5-\sqrt{33}}{2} \end{bmatrix}$$

These rows are the same relation, but again it isn't obvious. We can use either row to form the linear relationship. $x + \frac{5-\sqrt{33}}{2}y$, or $y = \frac{-2}{5-\sqrt{33}}x$. So, the eigenvector is $\vec{v}_2 = \begin{bmatrix} 1 \\ \frac{-2}{5-\sqrt{33}} \end{bmatrix}$.

Then the second term of our solution is $\vec{x}_2(t) = e^{(3+\sqrt{33})t/2} \begin{bmatrix} 1 \\ \frac{-2}{5-\sqrt{33}} \end{bmatrix}$.

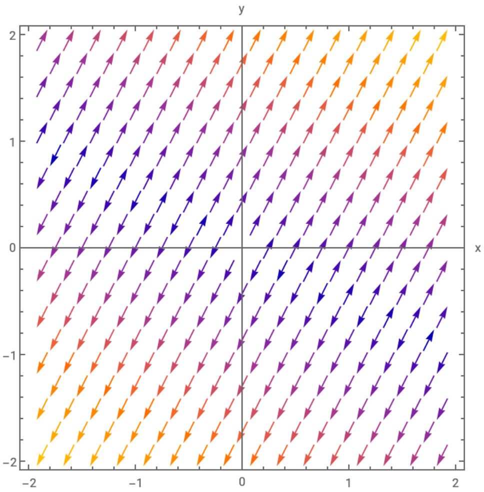
Finally, we combine the two solutions to form the general solution for $\vec{x}' = A\vec{x}$, $\vec{x}(t) = C_1 e^{(3-\sqrt{33})t/2} \begin{bmatrix} 1 \\ \frac{-2}{5+\sqrt{33}} \end{bmatrix} + C_2 e^{(3+\sqrt{33})t/2} \begin{bmatrix} 1 \\ \frac{-2}{5-\sqrt{33}} \end{bmatrix}$. ■

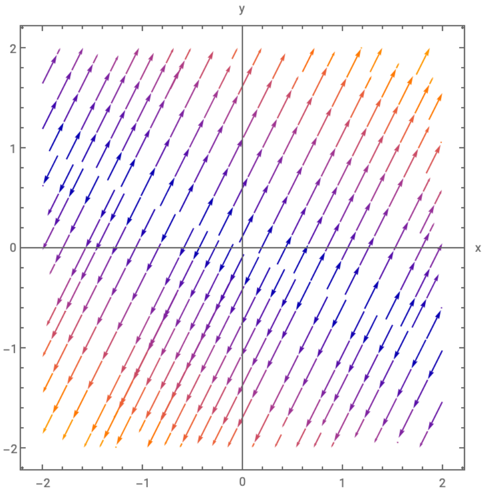
■

Let's look at the special case when one eigenvalue is zero, as we lead into the next section:

■ **Example 3.17** Determine the general solution for the linear system $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

This matrix is singular, $1*4-2*2 = 0$, so this is a unique case of solutions. When the matrix is singular, there is a *line* of equilibrium values, all solutions trend toward or away from that line in the phase plane. We can see that in the direction field and streamplot in Mathematica.





We'll still start by solving for the eigenvalues of A.

$A-\lambda I=\begin{bmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{bmatrix}$

$\det(A-\lambda I)=(1-\lambda)(4-\lambda)-4=4-5\lambda+\lambda^2-4=\lambda^2-5\lambda=\lambda(\lambda-5)=0$

$\lambda_1=0$, and $\lambda_2=5$.

Find the associated eigenvectors:

For $\lambda_1=0$, $A-\lambda_1 I=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

We can use the top equation, $x+2y=0$, to define a relationship $y=-1/2x$.

$\vec{v}_1=\begin{bmatrix}1\\ -1/2\end{bmatrix}$

For $\lambda_2=5$, $A-\lambda_2 I=\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}$

We can use the bottom equation, $2x-y=0$, to define a relationship $y=2x$.

$\vec{v}_2=\begin{bmatrix}1\\ 2\end{bmatrix}$

Then, we can write the general solution as $\vec{x}(t)=C_1e^{0t}\begin{bmatrix}1\\ -1/2\end{bmatrix}+C_2e^{5t}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}C_1+C_2e^{5t}\\ -1/2C_1+2C_2e^{5t}\end{bmatrix}$.

■

3.3 Phase Plane Analysis of Planar Systems

We can use the form of solution to define the phase portrait behavior - and we can use the phase portrait behavior to describe the nature of our solution.

The eigenvalues determine the stability of the equilibrium value, and the eigenvectors determine the general shape of the solutions in the phase plane.

There are three cases for distinct, real eigenvalues:

1) Two positive eigenvalues

2) Two negative eigenvalues

3) One positive eigenvalue and one negative eigenvalue

The sign of the eigenvalue determines whether the solution curve trends toward (negative) the equilibrium point, or away (positive) from it.

In the first case, both eigenvalues are positive so all solutions trend away from the equilibrium point. This case has an unstable node for an equilibrium point.

In the second case, both eigenvalues are negative so all solutions trend toward the equilibrium point. This case has an asymptotically stable node which is also called a sink.

For both of the first two cases, when the eigenvalues are the same sign, the eigenvalue with the greatest magnitude is referred to as the dominant eigenvalue. The solution curves will tend to follow the behavior of the dominant eigenvalue more closely in these cases.

In the third case, the eigenvalues have different signs. One solution is approaching the equilibrium, while the other moves away from it. The eigenvalue that is positive dominates the negative eigenvalue in this case, regardless of size. The equilibrium point is an unstable saddle. The only solution which approaches the equilibrium point is given along the line of the associated eigenvector for the negative eigenvalue. All other solution curves will move away from the equilibrium point.

In all cases, there are straight line solutions that follow the eigenvectors associated with the solution. The arrows (and thus the solutions) move toward the equilibrium point for negative eigenvalues, and away from the equilibrium point for positive eigenvalues.

We can use these relationships in both directions: If the eigenvalue is positive, we sketch in a line along its eigenvector and point the arrows away from the equilibrium point.

Also, if we see a line of arrows pointing away from the equilibrium value - we know that the eigenvalue associated with that solution is positive.

We can use the same reasoning for negative eigenvalues.

■ **Example 3.18** Sketch the direction field in the phase plane for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. Then, write the form of the general solution.

Determine the eigenvalues: $\det(A-\lambda I) = (2-\lambda)(3-\lambda) - 1 = \lambda^2 - 5\lambda + 5 = 0$

$\lambda = \dfrac{5 \pm \sqrt{25-4*1*5}}{2*1}$

$\lambda_1 = \dfrac{5-\sqrt{5}}{2} \approx 1.38197$

$\lambda_2 = \dfrac{5+\sqrt{5}}{2} \approx 3.61803$

Both eigenvalues are positive.

Determine the eigenvectors: $A - \lambda I$

For λ_1: $A - \lambda_1 I = \begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & 1 \\ 1 & \dfrac{1+\sqrt{5}}{2} \end{bmatrix}$

So the associated eigenvector is given by $\vec{v}_1 = \begin{bmatrix} 1 \\ \dfrac{-2}{1+\sqrt{5}} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.61803 \end{bmatrix}$

For λ_2: $A - \lambda_2 I = \begin{bmatrix} \dfrac{-1-\sqrt{5}}{2} & 1 \\ 1 & \dfrac{1-\sqrt{5}}{2} \end{bmatrix}$

and the associated eigenvector is given by $\vec{v}_2 = \begin{bmatrix} 1 \\ \dfrac{-2}{1-\sqrt{5}} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1.61803 \end{bmatrix}$

Constructing the phase portrait:

We have nonzero eigenvalues, so this matrix is nonsingular. That means the only equilibrium point for this first order linear system of differential equations is at $(0,0)$.

Both eigenvalues are positive, so the equilibrium point is an unstable node or source. All arrows will point away from the equilibrium point in the phase plane.

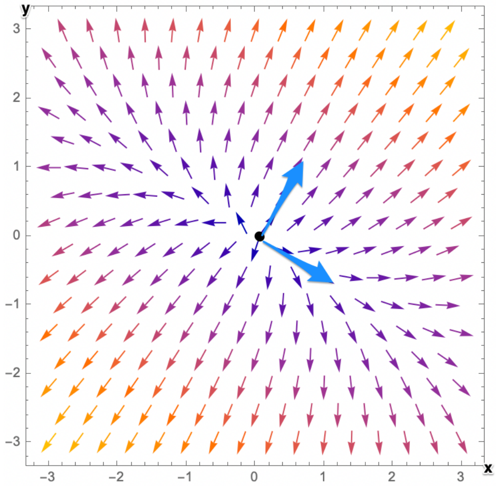
To sketch in the paths of the eigenvectors, it's helpful to reference the equations used to build them. $y = -0.61803x$ and $y = 1.61803x$ are the lines that are along the eigenvectors in this example (which is why I usually choose to define by eigenvectors by setting the $x = 1$ and determining the y

value through the relationship - it's easier to draw.)

You can still draw the direction field vectors using the values of the derivatives (just like before), but as we've mentioned before - that is time consuming to do by hand. The general picture that you need is defined by the eigenvalues and eigenvectors.

The larger eigenvalue is λ_2, so the vectors will all approach $\vec{v}_2$ in their end behavior. (λ_1 is much harder to spot in the graph.)

This can be seen in the direction field generated by Mathematica, with the eigenvectors approximated by the blue arrows.



The general form of solution is the same as in the previous section: $\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$.

So, for this problem that is $\vec{x}(t) = C_1 e^{(5-\sqrt{5})/2t}\begin{bmatrix}1\\ \dfrac{-2}{1+\sqrt{5}}\end{bmatrix} + C_2 e^{(5+\sqrt{5})/2t}\begin{bmatrix}1\\ \dfrac{-2}{1-\sqrt{5}}\end{bmatrix}$.

■

■ **Example 3.19** Sketch the direction field in the phase plane for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix}-2 & 3\\ 1 & -4\end{bmatrix}$.

Determine the eigenvalues: $\det(A-\lambda I) = (-2-\lambda)(-4-\lambda) - 3 = \lambda^2 + 6\lambda + 5 = 0$

$$\lambda = \frac{-6 \pm \sqrt{36 - 4*1*5}}{2*1}$$

$$\lambda_1 = \frac{-6-\sqrt{16}}{2} = -5$$

$$\lambda_2 = \frac{-6+\sqrt{16}}{2} = -1$$

Both eigenvalues are negative.

Determine the eigenvectors: $A - \lambda I$

For λ_1: $A - \lambda_1 I = \begin{bmatrix}3 & 3\\ 1 & 1\end{bmatrix}$

So the associated eigenvector is given by $\vec{v}_1 = \begin{bmatrix}1\\ -1\end{bmatrix}$

For λ_2: $A - \lambda_2 I = \begin{bmatrix}-1 & 3\\ 1 & -3\end{bmatrix}$

and the associated eigenvector is given by $\vec{v}_2 = \begin{bmatrix}1\\ 1/3\end{bmatrix}$

Constructing the phase portrait:

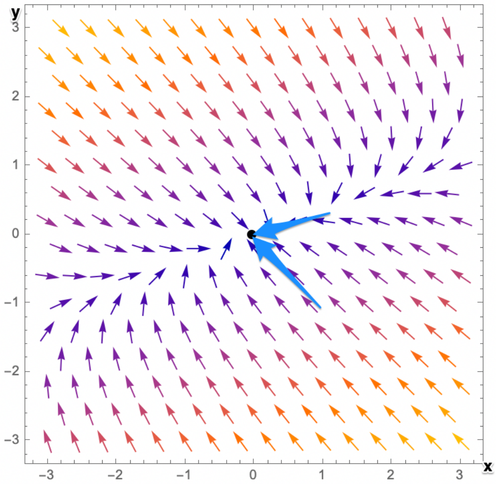
We have nonzero eigenvalues, so this matrix is nonsingular. That means the only equilibrium point for this first order linear system of differential equations is at $(0,0)$.

Both eigenvalues are negative, so the equilibrium point is an asymptotically stable node or sink. All arrows will point toward the equilibrium point in the phase plane.

To sketch in the paths of the eigenvectors, it's helpful to reference the equations used to build them. $y = -x$ and $y = 1/3x$ are the lines that are along the eigenvectors in this example.

The larger eigenvalue, by magnitude, is λ_1, so the vectors will all approach $\vec{v}_1$ in their end behavior. (λ_2 will be more visible in this case than the previous example.)

This can be seen in the direction field generated by Mathematica, with the eigenvectors approximated by the blue arrows.



The general solution is of the form $\vec{x}(t) = C_1 e^{-5t}\begin{bmatrix}1\\-1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1\\1/3\end{bmatrix}$. ■

■ **Example 3.20** Sketch the direction field in the phase plane for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix}2 & 0\\1 & -4\end{bmatrix}$.

Determine the eigenvalues: $\det(A-\lambda I) = (2-\lambda)(-4-\lambda) - 0 = \lambda^2 + 2\lambda - 8 = 0$ (this is factorable, so we can see that the values will be 2 and -4.)

$$\lambda = \frac{-2 \pm \sqrt{4 - 4*1*(-8)}}{2*1}$$
$$\lambda_1 = \frac{-2-\sqrt{36}}{2} = -4$$
$$\lambda_2 = \frac{-2+\sqrt{36}}{2} = 2$$

The first eigenvalue is negative, the second is positive.

Determine the eigenvectors: $A - \lambda I$

For λ_1: $A - \lambda_1 I = \begin{bmatrix}6 & 0\\1 & 0\end{bmatrix}$

So the associated eigenvector is given by $\vec{v}_1 = \begin{bmatrix}0\\1\end{bmatrix}$ This one is special, the equation is $x = 0$, so y is arbitrary. The simplest expression of this vector is to use the unit vector for y.

For λ_2: $A - \lambda_2 I = \begin{bmatrix}0 & 0\\1 & -6\end{bmatrix}$

and the associated eigenvector is given by $\vec{v}_2 = \begin{bmatrix}1\\1/6\end{bmatrix}$ This one is also special, we have to use the second row to define the eigenvector because the first row is all zeroes.

Constructing the phase portrait:

We have nonzero eigenvalues, so this matrix is nonsingular. That means the only equilibrium point for this first order linear system of differential equations is at $(0,0)$.

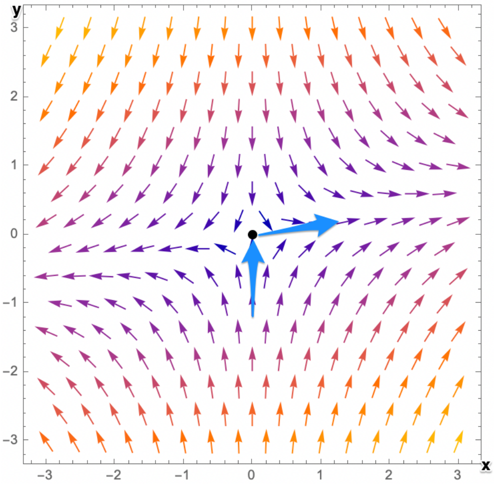
The eigenvalues are opposite signs, so the equilibrium point is an unstable saddle. There will be some arrows pointing towards the equilibrium point, but nearly all solutions will move away from the equilibrium point overall.

To sketch in the paths of the eigenvectors, it's helpful to reference the equations used to build them. $x = 0$ (special case when one of our variables has a zero coefficient in the analysis) and $y = 1/6x$ are the lines that are along the eigenvectors in this example.

You can still draw the direction field vectors using the values of the derivatives (just like before), but as we've mentioned before - that is time consuming to do by hand. The general picture that you need is defined by the eigenvalues and eigenvectors.

The positive eigenvalue is λ_2, so the vectors will all approach $\vec{v}_2$ in their end behavior. (λ_1 is larger, but negative. The behavior will trend toward the positive eigenvalue in these problems.)

This can be seen in the direction field generated by Mathematica, with the eigenvectors approximated by the blue arrows.



The general solution is of the form $\vec{x}(t) = C_1 e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1/6 \end{bmatrix}$.

■

3.4 Complex Eigenvalues

Complex eigenvalues cause some amount of "swirl" to the graph. This is due to the nature of the solutions that are formed by complex eigenvalues. Recall that for real eigenvalues, our solutions were of the form $C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2$. Systems with complex eigenvalues have solutions of the same general form, but we have to address what it means for an exponential to have a complex exponent.

We need to use Euler's formula: $e^{i\beta t} = \cos(\beta t) + i \sin(\beta t)$

This introduces the oscillating behavior we described in the mass-spring problems. An under-damped oscillator contains real and complex parts, so it oscillates, but the damping term causes it to ultimately match the real part of its eigenvalue. An undamped oscillator contains only complex parts, and oscillates indefinitely.

As you may recall from trigonometry, the oscillations in (t, x) and (t, y) space translate to circles in (x, y) space (remember your unit circle?!)

This is the behavior that causes what I am referring to as a "swirl" in the phase plane. The eigenvectors no longer assist us in the graph, because the eigenvectors are complex, and any notion of a direction for trajectories is lost. However, they are still relevant in defining the solutions to these problems. Let's do a few examples.

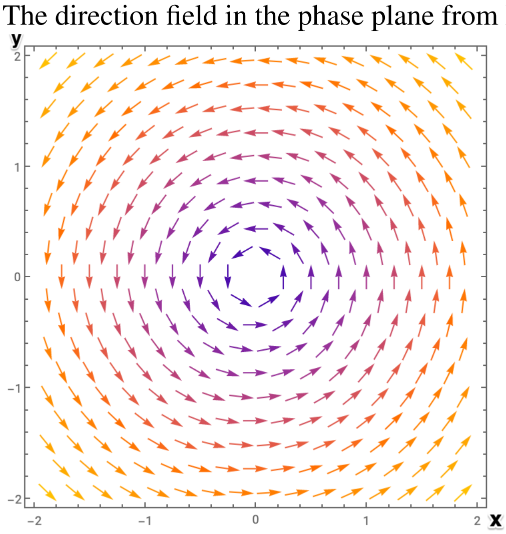
■ **Example 3.21** Sketch the direction field in the phase plane for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Then, determine the general form of solution, a particular solution for initial values $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and check the consistency of your solution with the direction field.

First, we determine the eigenvalues: $A - \lambda I = \begin{bmatrix} -\lambda & -2 \\ 2 & -\lambda \end{bmatrix}$
$\det(A - \lambda I) = \lambda^2 + 4 = 0.$
$\lambda_1 = -2i$
$\lambda_2 = 2i$
Then, we determine the eigenvectors:
For $\lambda_1 = -2i$: $A - \lambda_1 I = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}$
Using the first equation $2ix - 2y = 0$, we can define a relationship $y = ix$.
$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$
For $\lambda_2 = 2i$: $A - \lambda_2 I = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$
Using the first equation $-2ix - 2y = 0$, we can define a relationship $y = -ix$
$\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Due to the form of the eigenvalues (purely complex), the resulting direction field results in circles. The circles in the phase plane oscillate about the equilibrium point at $(0,0)$. We then refer to the equilibrium point as a **stable center** (all solutions neither trend toward or away from the equilibrium point, they oscillate about it.)

The direction field in the phase plane from Mathematica shows the circular behavior.



The general solution changes.
$\vec{x}(t) = C_1 e^{-2it} \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{2it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

These two solutions are conjugates of each other, so the resulting general solution is only truly dependent on the form of the eigenvalues.

Using Euler's formula, this changes chancges $e^{2it} = \cos(2t) + i\sin(t)$, since $e^{-2it} = \cos(2t) - i\sin(t)$

$\vec{x}(t) = C_1(\cos(2t) - i\sin(t)) \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2(\cos(2t) + i\sin(t)) \begin{bmatrix} 1 \\ -i \end{bmatrix}$

This causes some issues because it looks like $\vec{x}(t)$ is going to be a complex function... it can't be.

Instead, we break $\vec{x}$ into its real components and its imaginary components.

$\vec{x}(t) = \begin{bmatrix} C_1(\cos(2t) - i\sin(2t)) + C_2(\cos(2t) + i\sin(2t)) \\ C_1 i(\cos(2t) - i\sin(2t)) + C_2(-i)(\cos(2t) + i\sin(2t)) \end{bmatrix} = \begin{bmatrix} (C_1 + C_2)\cos(2t) + (C_2 - C_1)i\sin(2t) \\ (C_1 - C_2)i\cos(2t) + (C_1 + C_2)\sin(2t) \end{bmatrix}$

More generally, we write the solutions of the form

$\vec{x}(t) = ARe[\vec{x}(t)] + BIm[\vec{x}(t)] = A\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + B\begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}$ in this example, where A and B are such that the resulting function is real.

Alternatively, we can represent both e^{2it} and e^{-2it} through a general solution based on one of the two conjugates, and define the two solutions by the real and imaginary parts of that solution. We will do that now using e^{2it} to construct the general solution. $e^{2it} = \cos(2t) + i\sin(2t)$ (this is using λ_2, so we will call the solution term $\vec{x}_2(t)$

$\vec{x}_2(t) = (\cos(2t) + i\sin(2t))\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos(2t) + i\sin(2t) \\ -i\cos(2t) + \sin(2t) \end{bmatrix}$

Separating it into the Real ($Re[]$) and Imaginary ($Im[]$) components:

$Re[\vec{x}_2(t)] = \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$

$Im[\vec{x}_2(t)] = \begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}$

We arrive at the same form we had above (with less work):

$\vec{x}(t) = A\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + B\begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix} = \begin{bmatrix} A\cos(2t) + B\sin(2t) \\ A\sin(2t) - B\cos(2t) \end{bmatrix}$

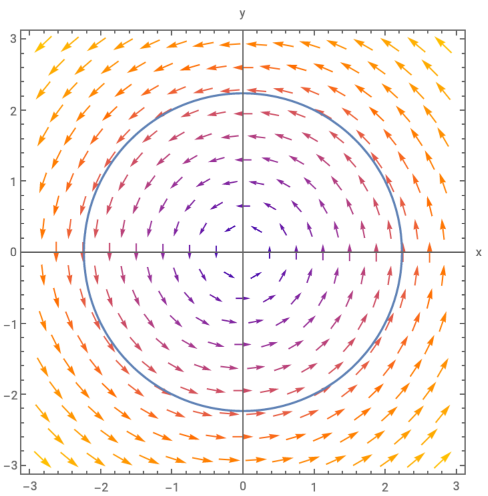
Then, we can apply an initial condition, like $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

To do that, we do as we normally do - because we've removed any imaginary parts from the form of general solution.

$\vec{x}(0) = \begin{bmatrix} A + 0 \\ 0 - B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So, $A = 1$, $B = -2$, and $\vec{x}(t) = \begin{bmatrix} \cos(2t) - 2\sin(2t) \\ \sin(2t) + 2\cos(2t) \end{bmatrix}$

You can verify the consistency of your solution by plotting a parametric curve using $x(t) = \cos(2t) - 2\sin(2t)$ and $y(t) = \sin(2t) + 2\cos(2t)$.



■

■ **Example 3.22** Sketch the direction field in the phase plane for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

Then, determine the general form of solution, a particular solution for initial values $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and check the consistency of your solution with the direction field.

First, we determine the eigenvalues: $A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix}$

$\det(A-\lambda I) = (1-\lambda)^2 + 4 = 1 - 2\lambda + \lambda^2 + 4 = \lambda^2 - 2\lambda + 5 = 0.$

$\lambda_1 = \dfrac{2 - \sqrt{4-4(1)(5)}}{2} = 1 - 2i$

$\lambda_2 = \dfrac{2 + \sqrt{4-4(1)(5)}}{2} = 1 + 2i$

Then, we determine the eigenvectors:

For $\lambda_1 = 1-2i$: $A - \lambda_1 I = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}$

Using the first equation $(2i)x - 2y = 0$, we can define a relationship $y = ix$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

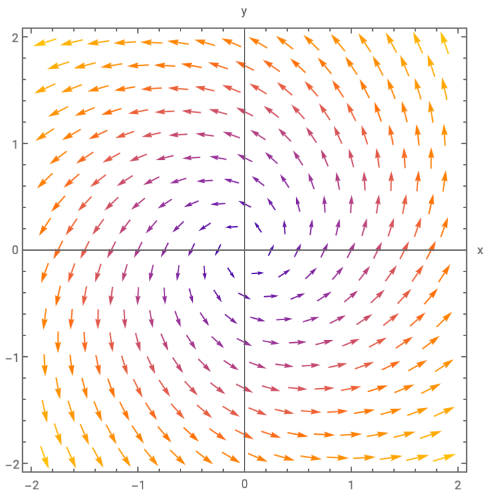
For $\lambda_2 = 1+2i$: $A - \lambda_2 I = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$

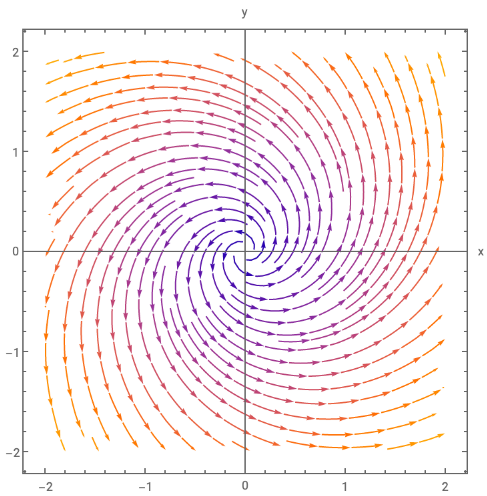
Using the first equation $(-2i)x - 2y = 0$, we can define a relationship $y = -ix$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Due to the form of the eigenvalues (parts that are real and complex), the resulting direction field results in spirals. The spirals in the phase plane curve out of the equilibrium point at $(0,0)$ (this is because the real part of our eigenvalues is positive, which is consistent with the analysis in the previous section.) This is referred to as a **spiral source**.

The direction field in the phase plane from Mathematica shows the spiraling behavior. The streamplot is a nice visual, so it is also included here.





The general solution will be a little more complicated than the last example. $\vec{x}(t) = C_1 e^{(1-2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{(1+2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

However, we learned in the last example that we can use one of the two solutions ($\vec{x}_1(t)$ or $\vec{x}_2(t)$) and construct the general form of solution from the Real and Imaginary parts of that solution (due to the conjugate form of our eigenvalues and eigenvectors.)

This time, let's use $\vec{x}_1(t) = e^{(1-2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$

$e^{(1-2i)t} = e^t \left(\cos(2t) - i\sin(2t)\right)$

$\vec{x}_1(t) = e^t \begin{bmatrix} \cos(2t) - i\sin(2t) \\ i\cos(2t) + \sin(2t) \end{bmatrix}$

Separating the Real and Imaginary parts:

$Re[\vec{x}_1(t)] = e^t \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$

$Im[\vec{x}_1(t)] = e^t \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}$

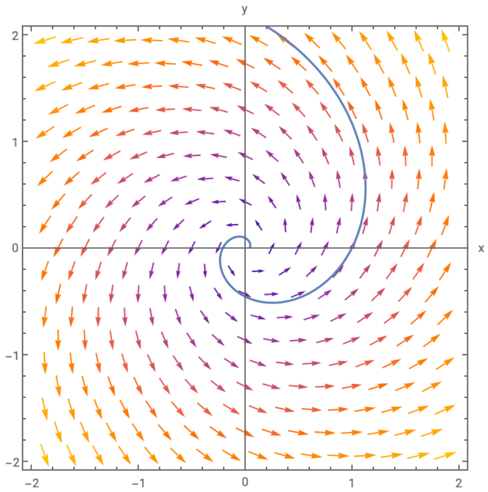
So, the general solution is $\vec{x}(t) = Ae^t \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + Be^t \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} = e^t \begin{bmatrix} A\cos(2t) - B\sin(2t) \\ A\sin(2t) + B\cos(2t) \end{bmatrix}$, where A and B are real.

The particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

$\vec{x}(0) = e^0 \begin{bmatrix} A-0 \\ 0+B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $A = 1$ and $B = 0$.

The particular solution is then $\vec{x}(t) = e^t \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$.

We can verify the solution using a parametric plot of $x(t) = e^t\cos(2t)$ and $y(t) = e^t\sin(2t)$.



■

■ **Example 3.23** Sketch the direction field in the phase plane for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$. Then, determine the general form of solution, a particular solution for initial values $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and check the consistency of your solution with the direction field.

First, we determine the eigenvalues: $A - \lambda I = \begin{bmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (-1-\lambda)^2 + 4 = 1 + 2\lambda + \lambda^2 + 4 = 0$

$\lambda_1 = \dfrac{-2 - \sqrt{4 - 4(1)(5)}}{2} = -1 - 2i$

$\lambda_2 = \dfrac{-2 + \sqrt{4 - 4(1)(5)}}{2} = -1 + 2i$

Then, we determine the eigenvectors:

For $\lambda_1 = -1 - 2i$: $A - \lambda_1 I = \begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}$

Using the first equation, $(2i)x - 2y = 0$, we can define a relationship $y = ix$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

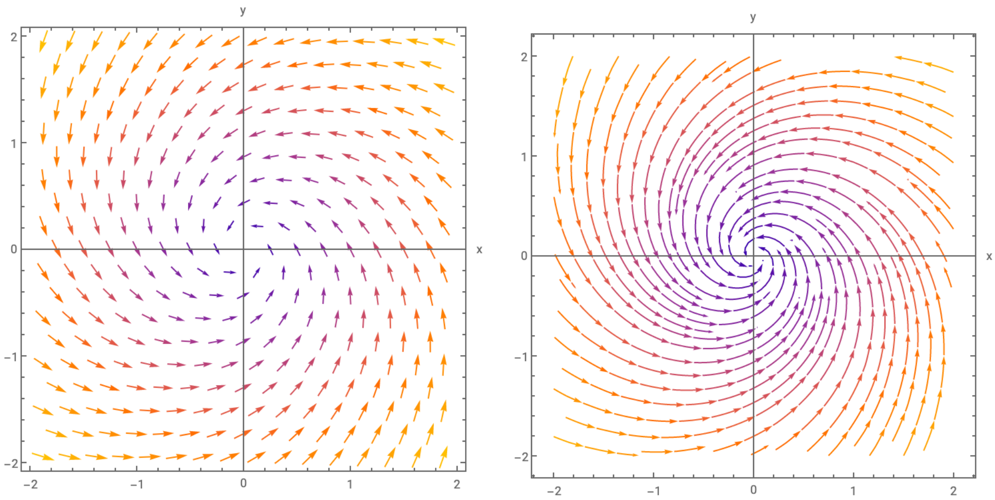
For $\lambda_2 = -1 + 2i$: $A - \lambda_2 I = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$

Using the first equation, $(-2i)x - 2y = 0$, we can define a relationship $y = -ix$.

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Due to the form of the eigenvalues, real and imaginary parts, the resulting direction field results in spirals. The spirals in the phase plane curve into the equilibrium point at $(0,0)$ because the real part of the eigenvalues is negative (consistent with the previous section.) This is referred to as a **spiral sink**.

The direction field and streamplot from Mathematica show the spiraling behavior (this time into the origin!)



We know from the last two examples that we can select one of our solutions to define the general solution form. Let's use $\vec{x}_2(t) = e^{(-1+2i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Using Euler's formula, $e^{(-1+2i)t} = e^{-t}(\cos(2t) + i\sin(2t))$

$$\vec{x}_2(t) = e^{-t}\begin{bmatrix} \cos(2t) + i\sin(2t) \\ -i\cos(2t) + \sin(2t) \end{bmatrix}$$

Separating it into the Real and Imaginary parts:

$$Re[\vec{x}_2(t)] = e^{-t}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$$

$$Im[\vec{x}_2(t)] = e^{-t}\begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix}$$

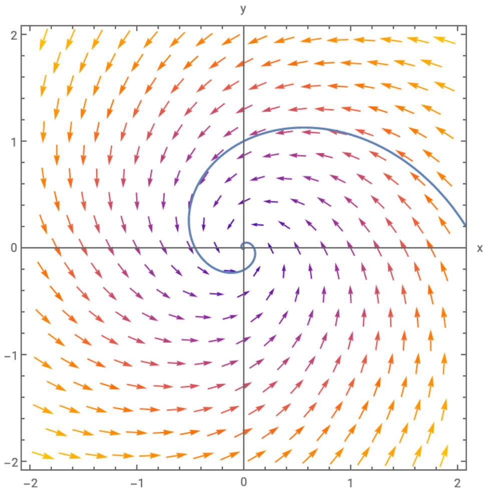
So, $\vec{x}(t) = Ae^{-t}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + Be^{-t}\begin{bmatrix} \sin(2t) \\ -\cos(2t) \end{bmatrix} = e^{-t}\begin{bmatrix} A\cos(2t) + B\sin(2t) \\ A\sin(2t) - B\cos(2t) \end{bmatrix}$, where A and B are real.

The particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$\vec{x}(0) = e^0\begin{bmatrix} A+0 \\ 0-B \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $A = 0$ and $B = -1$.

Then, $\vec{x}(t) = e^{-t}\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}$.

We can verify this solution using a parametric plot of $x(t) = -e^{-t}\sin(2t)$ and $y(t) = e^{-t}\cos(2t)$.



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3.5 Repeated Eigenvalues

Complex eigenvalues are fun, but repeated eigenvalues are tricky. We have repeated eigenvalues when there is only one eigenvalue. It does not occur frequently, but it does pose a problem when we need two, linearly independent, solutions to form a solution to a linear system.

When we only have one eigenvalue, then both solutions take the same form. So, in order to form a linearly independent solution, we introduce a t into the solution term, and have to determine a second eigenvector to be associated with that solution.

This is best articulated through an example.

■ **Example 3.24** Determine the direction field and general solution for $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$.

Determine the eigenvalues: $A - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ -1 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (2-\lambda)(2-\lambda) - 0 = 0$

So, $\lambda = 2$ is the only eigenvalue, and it is repeated. Hmm.

Well, we can find the associated eigenvector for $\lambda = 2$.

$A - \lambda I = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, as we learned in a previous example - $x = 0$ and $y =$anything.

So, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a natural choice for the associated eigenvector.

$\vec{x}_1(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

However, we need another eigenvector to construct two linearly independent solutions for this system, $\vec{x}_1(t)$ and $\vec{x}_2(t)$.

We can fall back on our solutions to partially coupled systems to solve this equation - and use the solution to define the method of solving for repeated eigenvalues more generally.

Expanding $\vec{x}' = A\vec{x}$ we can look at the individual equations.

$x' = 2x$ and $y' = -x + 2y$.

Okay, so $x(t) = C_1 e^{2t}$

Then, we can solve for $y(t)$ using integrating factor.

$y' - 2y = -C_1 e^{2t}$

$\mu(t) = e^{-2t}$

$(e^{-2t} y)' = -C_1$

$e^{-2t}y = -C_1t + C_2$

$y(t) = -C_1te^{2t} + C_2e^{2t}$

So, $\vec{x}(t) = \begin{bmatrix} C_1e^{2t} \\ -C_1te^{2t} + C_2e^{2t} \end{bmatrix} = C_1e^{2t}\begin{bmatrix} 1 \\ -t \end{bmatrix} + C_2e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The second term, $C_2e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ looks like what we got from the eigenvalue and its associated eigenvector. So, we need a method to arrive at the first term more generally.

To do this, we seek another eigenvector through solving $(A - \lambda I)\vec{v}_2 = \vec{v}_1$.

In this problem, that is $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which results in $x = -1$, and $y =$ anything.

In this case, that "anything" is represented by 0 because then it is clearly linearly independent from $\vec{v}_1$ (they are then both basis vectors, which span the space of $\mathcal{R}^2$.

$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\vec{x}_2(t) = e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$

$\vec{x}_2(t) = e^{2t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

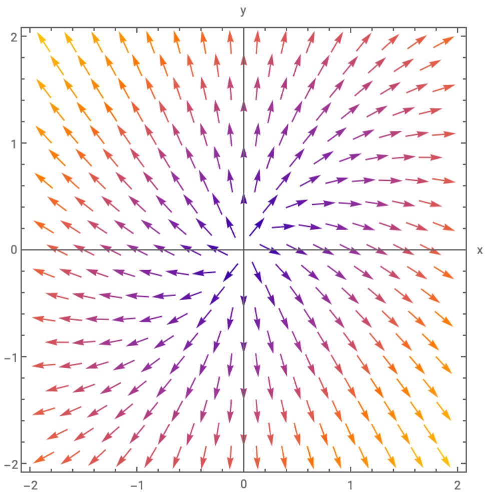
Then $\vec{x}(t) = A\vec{x}_1(t) + B\vec{x}_2$

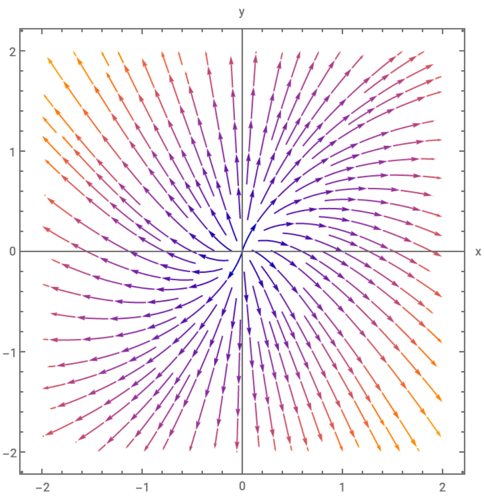
$\vec{x}(t) = Ae^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + Be^{2t}\begin{bmatrix} -1 \\ t \end{bmatrix} = \begin{bmatrix} -Be^{2t} \\ Ae^{2t} + Bte^{2t} \end{bmatrix}$

This is consistent with our earlier solution, where $B = -C_1$ and $A = C_2$.

For reference, the earlier solution was: $\vec{x}(t) = \begin{bmatrix} C_1e^{2t} \\ -C_1te^{2t} + C_2e^{2t} \end{bmatrix}$

Huzzah! Let's look at the direction field.





Notice in both the direction field and the streamplot that the solutions have a slight twist - but there isn't a clear long-term behavior of the solution. Each solution moves away from the origin (positive eigenvalue), but doesn't align with any specific eigenvector (unlike the solutions with real, positive, distinct eigenvalues.) There does appear to be a straight-line solution along the y-axis, consistent with our original eigenvector for $\lambda = 2$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

You might hypothesize that the solutions will eventually align with that eigenvector, but they don't. If you zoom out, the picture remains the same - like a splatter of solution curves out of the origin. This is a distinction from the spiraling of the complex eigenvalues - there is not a spiral here and there is no oscillation in the solution. ■

Another example!

■ **Example 3.25** Determine the direction field in the phase plane, the general solution, a particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, and check the consistency of your solution in the phase plane for the solution of $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}$

First, we find the eigenvalues. $A - \lambda I = \begin{bmatrix} -3-\lambda & 0 \\ 1 & -3-\lambda \end{bmatrix}$

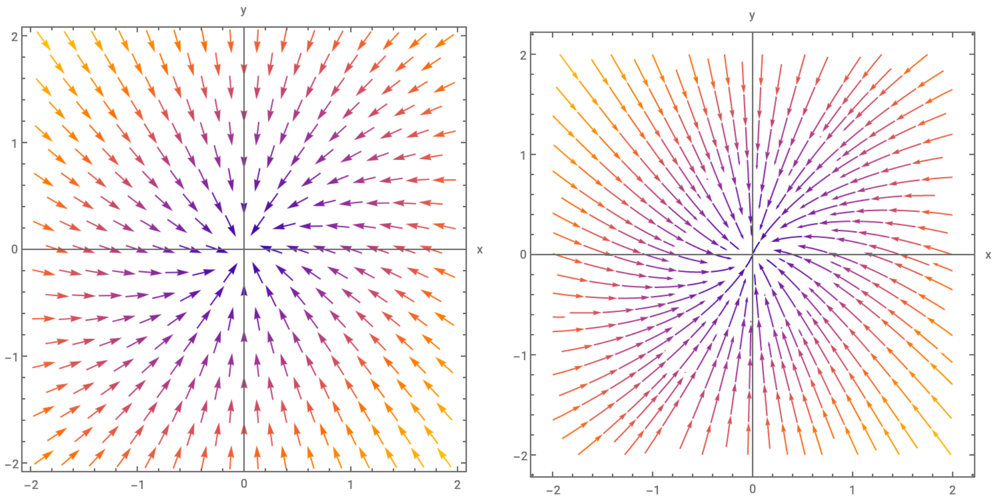
$\det(A - \lambda I) = (-3-\lambda)(-3-\lambda) - 0 = 0$

So, we have one eigenvalue, $\lambda = -3$, that is repeated.

Determine its associated eigenvector: $A - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

So, $x = 0$, $y =$ anything. So, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We know that the solutions will trend toward the origin because the eigenvalue is negative. Additionally, because it is repeated, we know it will be that "splatter" of solutions and have only a slight twist into the origin, along with a straight line solution along the eigenvector.



We can construct one of our linearly independent solutions using λ and $\vec{v}$ as usual: $\vec{x}_1(t) = e^{-3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

To determine a second solution we need a second eigenvector, we solve for $(A - \lambda I)\vec{v}_2 = \vec{v}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In this, $x = 1$, and $y =$ anything - but to make the eigenvectors linearly independent we use $y = 0$ (two basis vectors.)

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then, we construct the second solution $\vec{x}_2(t) = e^{\lambda t}(\vec{v}_2 + t\vec{v}_1)$

$\vec{x}_2(t) = e^{-3t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

So, the general solution $\vec{x}(t) = A\vec{x}_1(t) + B\vec{x}_2(t)$

$= Ae^{-3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + Be^{-3t} \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} Be^{-3t} \\ Ae^{-3t} + Bte^{-3t} \end{bmatrix}$

The particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$:

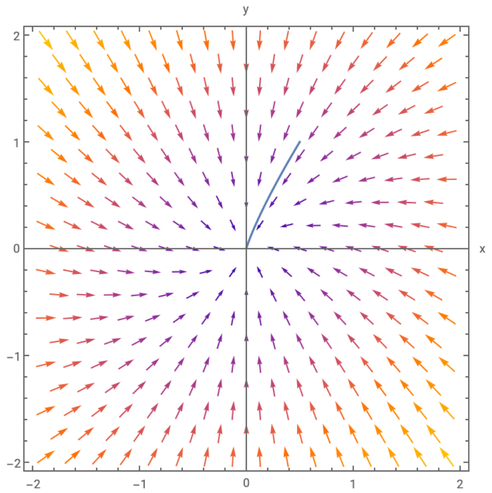
Substitute $t = 0$ into $\vec{x}(t)$

$\vec{x}(0) = \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

So, $A = 1$ and $B = 1/2$

The particular solution is $\vec{x}(t) = \begin{bmatrix} 1/2e^{-3t} \\ e^{-3t} + 1/2te^{-3t} \end{bmatrix}$

We can verify its consistency by plotting the curve on the same axes as the direction field in the phase plane. $x(t) = 1/2e^{-3t}$ and $y(t) = e^{-3t} + 1/2te^{-3t}$. (Note: this is plotted from $t = 0$ and forward in time, so you see the point it passes through at $t = 0$, and then that the curve moves toward the equilibrium point at $(0,0)$.)



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Let's do one that is less obvious.

■ **Example 3.26** Determine the direction field in the phase plane, the general solution, a particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1 \\ 2/5 \end{bmatrix}$, and check the consistency of your solution in the phase plane for the solution of $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}$

First, as always, find the eigenvalues of A. $A - \lambda I = \begin{bmatrix} -4-\lambda & 3 \\ -3 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (-4-\lambda)(2-\lambda) + 9 = -8 + 2\lambda + \lambda^2 + 9 = \lambda^2 + 2\lambda + 1 = 0$

This is factorable, $(\lambda + 1)^2 = 0$, and it is also the case when the quadratic formula term $b^2 - 4ac = 4 - 4(1)(1) = 0$.

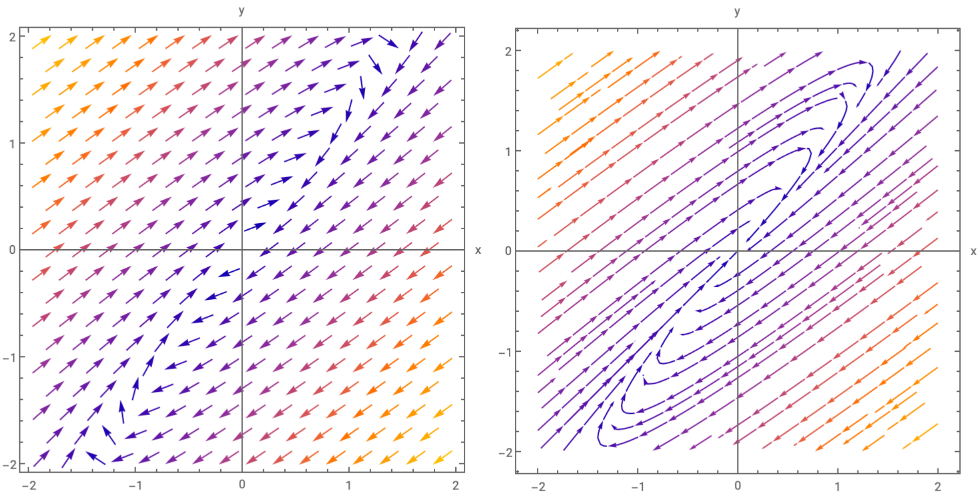
We have a repeated eigenvalue $\lambda = -1$.

Find its associated eigenvector: $A - \lambda I = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}$

So, the eigenvector needs to have the relationship $-3x + 3y = 0$, or $y = x$.

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We know (roughly) what the direction field will look like. We have a repeated eigenvalue that is negative, so it's a splatter of solutions into the origin with a straight line solution along the eigenvector. Verifying with Mathematica plots:



When we move off the axes (our previous two examples were “simple" cases, with basis vectors), we see the “twist" of the solutions to align with the eigenvector. It looks different from the previous cases - but it’s a consistent appearance. All the solutions are aligning with one eigenvector.

Let’s construct the solutions.

The first solution term is given by $\vec{x}_1(t) = e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$

For the second solution, we seek a second eigenvector, $\begin{bmatrix}-3 & 3\\-3 & 3\end{bmatrix}\vec{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

So, the second eigenvector needs to maintain the relationship $-3x+3y=1$, or $y = x+1/3$

$\vec{v}_2 = \begin{bmatrix}1\\4/3\end{bmatrix}$

An important check: make sure the eigenvectors are linearly independent: construct a matrix using the eigenvectors and double check that the matrix is not singular.

$V = \begin{bmatrix}\vec{v} & \vec{v}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 4/3\end{bmatrix}$, the $\det(V) = 4/3-1 = 1/3 \neq 0$ (Great! Our eigenvalues are linearly independent, we can use $\vec{v}_2$.)

Then, the second solution is $\vec{x}(t) = e^{-t}\left(\begin{bmatrix}1\\4/3\end{bmatrix}+t\begin{bmatrix}1\\1\end{bmatrix}\right) = e^{-t}\begin{bmatrix}1+t\\4/3+t\end{bmatrix}$

Our general solution is then $\vec{x}(t) = Ae^{-t}\begin{bmatrix}1\\1\end{bmatrix}+Be^{-t}\begin{bmatrix}1+t\\4/3+t\end{bmatrix} = \begin{bmatrix}Ae^{-t}+Be^{-t}+Bte^{-t}\\Ae^{-t}+4/3Be^{-t}+Bte^{-t}\end{bmatrix}$

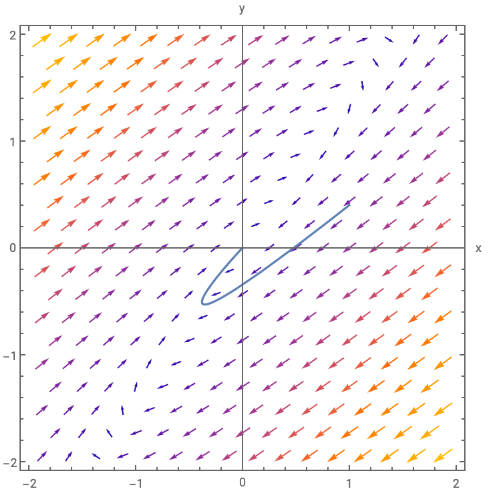
The particular solution satisfying $\vec{x}(0) = \begin{bmatrix}1\\2/5\end{bmatrix}$:

$\vec{x}(0) = \begin{bmatrix}A+B+0\\A+4/3B+0\end{bmatrix} = \begin{bmatrix}1\\2/5\end{bmatrix}$

So, $A+B=1$, $A+4/3B = 2/5$, so if we subtract equation 1 from equation 2: $1/3B = -3/5$, or $B = -9/5$, so $A = 14/5$.

Then, the particular solution is $\vec{x}(t) = \begin{bmatrix}14/5e^{-t}-9/5e^{-t}-9/5te^{-t}\\14/5e^{-t}-4/3(9/5)e^{-t}-9/5te^{-t}\end{bmatrix} = \begin{bmatrix}e^{-t}-9/5te^{-t}\\2/5e^{-t}-9/5te^{-t}\end{bmatrix}$.

Plotting this solution in the phase plane, we can check the consistency of our solution. $x(t) = e^{-t}-9/5te^{-t}$ and $y(t) = 2/5e^{-t}-9/5te^{-t}$.



■

Extra example, for additional context.

■ **Example 3.27** Determine the direction field in the phase plane, the general solution, a particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and check the consistency of your solution in the phase plane for the solution of $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$

First, as always, find the eigenvalues of A. $A - \lambda I = \begin{bmatrix} 4-\lambda & 3 \\ -3 & -2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (4-\lambda)(-2-\lambda) + 9 = -8 - 2\lambda + \lambda^2 + 9 = \lambda^2 - 2\lambda + 1 = 0$

This is factorable, $(\lambda - 1)^2 = 0$, and it is also the case when the quadratic formula term $b^2 - 4ac = 4 - 4(1)(1) = 0$.

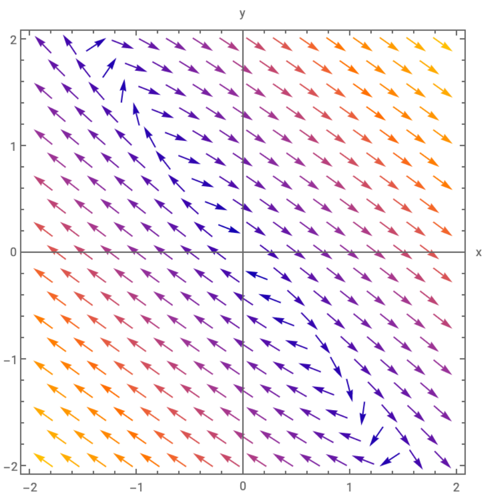
We have a repeated eigenvalue $\lambda = 1$.

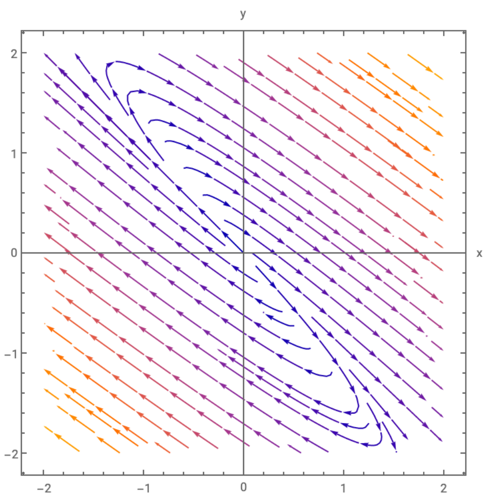
Find its associated eigenvector: $A - \lambda I = \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$

So, the eigenvector needs to have the relationship $3x + 3y = 0$, or $y = -x$.

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

We know (roughly) what the direction field will look like. We have a repeated eigenvalue that is positive, so it's a splatter of solutions away from the origin with a straight line solution along the eigenvector. Verifying with Mathematica plots:





When we move off the axes (our first two examples were “simple" cases, with basis vectors), we see the “twist" of the solutions to align with the eigenvector. It looks different from the previous cases - but it’s a consistent appearance. All the solutions are aligning with one eigenvector.

Let’s construct the solutions.

The first solution term is given by $\vec{x}_1(t) = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For the second solution, we seek a second eigenvector, $\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, the second eigenvector needs to maintain the relationship $3x+3y=1$, or $y=-x+1/3$

$\vec{v}_2 = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$

An important check: make sure the eigenvectors are linearly independent: construct a matrix using the eigenvectors and double check that the matrix is not singular.

$V = \begin{bmatrix} \vec{v} & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2/3 \end{bmatrix}$, the $\det(V) = -2/3-1 = -5/3 \neq 0$ (Great! Our eigenvalues are linearly independent, we can use $\vec{v}_2$.)

Then, the second solution is $\vec{x}(t) = e^t \left(\begin{bmatrix} 1 \\ -2/3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = e^t \begin{bmatrix} 1+t \\ -2/3-t \end{bmatrix}$

Our general solution is then $\vec{x}(t) = Ae^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + Be^t \begin{bmatrix} 1+t \\ -2/3-t \end{bmatrix} = \begin{bmatrix} Ae^t + Be^t + Bte^t \\ -Ae^t - 2/3Be^t - Bte^t \end{bmatrix}$

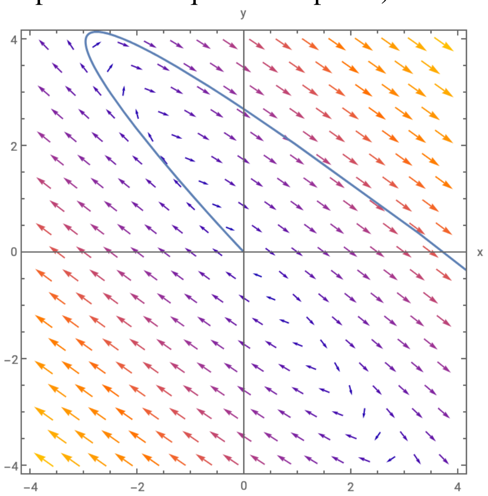
The particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$\vec{x}(0) = \begin{bmatrix} A+B+0 \\ -A-2/3B+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So, $A+B=1$, $-A-2/3B=2$, so if we add the two equations together: $1/3B=3$, or $B=9$, so $A=-8$.

Then, the particular solution is $\vec{x}(t) = \begin{bmatrix} -8e^t + 9e^t + 9te^t \\ 8e^t - 2/3(9)e^t - 9te^t \end{bmatrix} = \begin{bmatrix} e^t + 9te^t \\ 2e^t - 9te^t \end{bmatrix}$.

Plotting this solution in the phase plane, we can check the consistency of our solution. $x(t) = e^t + 9te^t$ and $y(t) = 2e^t - 9te^t$. (Note: this one includes negative values of time in order to include the shape near the equilibrium point.)



■

3.6 The Trace-Determinant Plane

This section is 3.7 in the text (we are skipping 3.6).

The Trace-Determinant plane is a summary of our work in Chapter 3. We can use the trace and determinant of our matrix to determine the expected behavior of our solutions.

Essentially, we noticed that the values in our matrix ultimately define the characteristic equation from the values of the trace and determinant of our matrix.

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, when we define the eigenvalues, we calculate the determinant of $A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0$

The $\det(A) = ad - bc$, and the trace of A ($tr(A)$) is $a+d$.

So, the relationship from our quadratic formula inside the square root is $(a+d)^2 - 4(1)(ad-bc)$ in terms of the terms of our matrix.

The possible cases:

1) Real eigenvalues: $(a+d)^2 - 4(1)(ad-bc) > 0$

2) Complex eigenvalues: $(a+d)^2 - 4(1)(ad-bc) < 0$

3) Repeated eigenvalue: $(a+d)^2 - 4(1)(ad-bc) = 0$

We can immediately determine the behaviors without doing all the analysis of the previous sections. To determine the solutions, we still need that analysis - but if we're simply trying to determine the behaviors, it's all in the values of our trace, T, and determinant,D, of A.

Further, the $tr(A) = \lambda_1 + \lambda_2$ and the $\det(A) = \lambda_1\lambda_2$. (This arises due to the form of the characteristic equation - if we factor the characteristic equation to determine the eigenvalues, this will always be true.)

Using these relationships we can further analyze the results.

1. If the eigenvalues are real, we have three (regular) cases for real eigenvalues, and a special case. Recall that $T^2 - 4D > 0$ to have real eigenvalues.
   (a) Both are positive, unstable source - occurs when both T and D are positive
   (b) Both are negative, asymptotically stable sink - occurs when T is negative but D is positive
   (c) One is positive and the other is negative, unstable saddle - occurs when D is negative
   (d) Special case - one eigenvalue is zero. This occurs when $D = 0$, resulting in a line of equilibrium points.
      i. The second eigenvalue is positive, unstable line of solution curves trending away from the line of equilibrium points. Occurs when $T > 0$.
      ii. The second eigenvalue is negative, asymptotically stable line of solution curves trending toward the line of equilibrium points. Occurs when $T < 0$.
2. If the eigenvalues are complex, we have another three cases. Recall that $T^2 - 4D < 0$ to have complex eigenvalues.
   (a) Purely imaginary eigenvalues (no real part), stable center - occurs when $T = 0$ and $D > 0$.
   (b) Positive real part, unstable spiral source - occurs when $T > 0$
   (c) Negative real part, asymptotically stable spiral sink - occurs when $T < 0$
3. If there is one eigenvalue that is repeated, we have two possible cases. Recall that $T^2 - 4D = 0$ for repeated eigenvalues.
   (a) The eigenvalue is positive, unstable source - occurs when $T > 0$
   (b) The eigenvalue is negative, asymptotically stable sink - occurs when $T < 0$

Understanding this plane is also helpful in analyzing bifurcations. If you're interested in exploring bifurcations further, take a class in Chaos and Dynamical Systems.

■ **Example 3.28** Determine the behavior of the solutions to $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$

Find the trace T and the determinant D:
$T = 1 + 3 = 4$
$D = 1 * 3 - 5 * (-2) = 3 + 10 = 13$
$T^2 - 4D = 4^2 - 4(13) = 16 - 52 < 0$ Complex eigenvalues.
$T > 0$, so the real part of the eigenvalues is positive.
Thus, our solutions will trend away from the origin, which is an unstable spiral source.
Verify: $\lambda_1 = 2 - 3i$, $\lambda_2 = 2 + 3i$.
$T = \lambda_1 + \lambda_2 = 4$ ✓
$D = \lambda_1 \lambda_2 = (2 - 3i)(2 + 3i) = 4 + 9 = 13$ ✓

■

■ **Example 3.29** Determine the behavior of the solutions to $\vec{x}' = A\vec{x}$ when $A = \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}$

Find the trace T and the determinant D:
$T = -3 + 2 = -1$
$D = (-3) * 2 - 5 * 1 = -6 - 5 = -11$
$T^2 - 4D = (-1)^2 - 4(-11) = 1 + 44 > 0$ Real eigenvalues.
$D < 0$, so the signs are different - one positive, one negative.
Thus, our solutions will trend away from the origin, excepting the one along the eigenvector for the negative eigenvalue. The equilibrium point is an unstable saddle.
Verify: $\lambda_1 = \dfrac{-1 - \sqrt{45}}{2}$, $\lambda_2 = \dfrac{-1 + \sqrt{45}}{2}$.
$T = \lambda_1 + \lambda_2 = -1$ ✓
$D = \lambda_1 \lambda_2 = \dfrac{-1 - \sqrt{45}}{2} \dfrac{-1 + \sqrt{45}}{2} = \dfrac{1 - 45}{4} = \dfrac{-44}{4} = -11$ ✓

■

Match the relationships as they are listed.

Go back through the examples from this chapter, and use the Trace-Determinant plane to define the behaviors we found the “long" way.

3.7 Linear Systems in Higher Dimensions

Solving the problem $\vec{x}' = A\vec{x}$ extends beyond the 2×2 case. We seek a solution of the form $e^{\lambda t}\vec{v}$, for any $n \times n$ system. Instead of 2 linearly independent solutions, we need n linearly independent solutions. However, these solutions are found in the same manner - determine the eigenvalues and eigenvectors, then take a linear combination of the solutions, $e^{\lambda t}\vec{v}$.

■ **Example 3.30** Determine the general solution for $\vec{x}' = A\vec{v}$ when $A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -5 & 1 \\ 0 & 1 & -4 \end{bmatrix}$

First, determine the eigenvalues:
$$A - \lambda I = \begin{bmatrix} -4-\lambda & 1 & 0 \\ 1 & -5-\lambda & 1 \\ 0 & 1 & -4-\lambda \end{bmatrix}$$
$\det(A - \lambda I) = (-4-\lambda)((-5-\lambda)(-4-\lambda) - (1)(1)) - 1(1(-4-\lambda) - (1)(0)) + 0(1(1) - (-5-\lambda)(0))$
$= (-4-\lambda)(20 + 9\lambda + \lambda^2 - 1) - (-4-\lambda)$
$= (-80 - 36\lambda - 4\lambda^2 + 4 - 20\lambda - 9\lambda^2 - \lambda^3 + \lambda) + 4 + \lambda$
$= -\lambda^3 - 13\lambda^2 - 54\lambda - 72 = 0$

Factors of 72: 1 and 72, 2 and 36, 3 and 24, 4 and 18, 6 and 12, 8 and 9 see if we can factor out any of these.

Evaluate $g(\lambda) = -\lambda^3 - 13\lambda^2 - 54\lambda - 72$ at the various values. It is clear that for positive λ values, we will never get a zero, but the negative values of possible factors will allow us to (hopefully) find one to factor and reduce the characteristic equation.

$g(1) = -140 \neq 0$

$g(-1) = 1 - 13 + 54 - 72 \neq 0$

$g(-2) = 8 - 13 * 4 + 54 * 2 - 72 = 116 - 124 \neq 0$

$g(-3) = 27 - 13 * 9 + 54 * 3 - 72 = 189 - 189 = 0$ Yay!

One of our factors is $(\lambda + 3)$ because the equation is zero at $\lambda = -3$.

Using long division, we can factor the characteristic polynomial to be $(\lambda + 3)(-\lambda^2 - 10\lambda - 24)$

We can then factor the remaining quadratic and write it as $-(\lambda + 3)(\lambda + 4)(\lambda + 6)$

Then, we find that our eigenvalues are $\lambda = -3, -4$, and -6.

To determine the associated eigenvectors, we proceed with the same process as our 2×2 systems: substitute in the eigenvalue and solve for the relationships.

For $\lambda_1 = -6$:

$$A - \lambda_1 I = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

So, the relationship $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

Leads to three possible equations: $2x + y = 0$, $x + y + z = 0$, or $y + 2z = 0$, what we need is a relationship that makes all three true (this was not a big deal in the 2×2 case because there were only two equations and they generally had the same relationship). From the first and third equations, we see that $y = -2x$ and $-2z$, so $x = z$, and $y = -2x$ is sufficient to form the eigenvector:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

You can double check the terms and see that all three equations are satisfied.

For $\lambda_2 = -4$:

$$A - \lambda_2 I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So, the relationship $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

Leads to three possible equations: $y = 0$, $x - y + z = 0$, and $y = 0$. Well, we know that $y = 0$, so then the second equation becomes $x + z = 0$ or $z = -x$. This is sufficient to construct the eigenvector:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda_3 = -3$:

$$A - \lambda_3 I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

So, the relationship $(A - \lambda_3 I)\vec{v}_3 = \vec{0}$

Leads to three possible equations: $-x + y = 0$, $x - 2y + z = 0$, and $y - z = 0$. So, $y = x$ and $y = z$, therefore $x = y = z$. That is enough to construct the eigenvector:

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We then construct the solution through $\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2 + C_3 e^{\lambda_3 t}\vec{v}_3$.

$$\vec{x}(t) = C_1 e^{-6t}\begin{bmatrix}1\\-2\\1\end{bmatrix} + C_2 e^{-4t}\begin{bmatrix}1\\0\\-1\end{bmatrix} + C_3 e^{-3t}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}C_1 e^{-6t} + C_2 e^{-4t} + C_3 e^{-3t}\\ -2C_1 e^{-6t} + C_3 e^{-3t}\\ C_1 e^{-6t} - C_2 e^{-4t} + C_3 e^{-3t}\end{bmatrix}$$

■

Complex cases also function similarly in 3×3 cases (and beyond.)

■ **Example 3.31** Determine the general solution for $\vec{x}' = A\vec{v}$ when $A = \begin{bmatrix}-2 & 2 & 1\\-2 & 2 & 2\\2 & -3 & -3\end{bmatrix}$

First, determine the eigenvalues:

$$A - \lambda I = \begin{bmatrix}-2-\lambda & 2 & 1\\-2 & 2-\lambda & 2\\2 & -3 & -3-\lambda\end{bmatrix}$$

$\det(A-\lambda I) = (-2-\lambda)((2-\lambda)(-3-\lambda)-(2)(-3)) - 2((-2)(-3-\lambda)-2(2)) + 1((-2)(-3) - (2-\lambda)(2))$

$= (-2-\lambda)(-6+\lambda+\lambda^2+6) - 2(6+2\lambda-4) + (6-4+2\lambda)$

$= (-2\lambda - 2\lambda^2 - \lambda^2 - \lambda^3) - (4+4\lambda) + (2+2\lambda)$

$= -\lambda^3 - 3\lambda^2 - 4\lambda - 2 = 0$

Factors of -2: 1, -1, 2, or -2.

Evaluate $g(\lambda) = -\lambda^3 - 3\lambda^2 - 4\lambda - 2$

$g(1) = -1-3-4-2 = -10 \neq 0$

$g(-1) = 1-3+4-2 = 0$ Yay!

We can factor out $\lambda + 1$

The resulting polynomial is $(\lambda+1)(-\lambda^2 - 2\lambda - 2)$

Using the quadratic formula we find the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1-i$, and $\lambda_3 = -1+i$.

Determine the associated eigenvectors.

For $\lambda_1 = -1$:

$$A - \lambda_1 I = \begin{bmatrix}-1 & 2 & 1\\-2 & 3 & 2\\2 & -3 & -2\end{bmatrix}$$

So, the relationship $(A-\lambda_1 I)\vec{v}_1 = \vec{0}$

Leads to three possible equations: $-x+2y+z=0$, $-2x+3y+2z=0$, and $2x-3y-2z=0$. These are more difficult to combine, but we can get there. Notice that equations 2 and 3 are the same, one is just negative of the other. So, we use the firs equation to say $x = 2y+z$ and substitute it into the second equation: $-2(2y+z)+3y+2z=0$, or $-y=0$, so $y=0$ and $x=z$.

$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

For $\lambda_2 = -1-i$:

$$A - \lambda_2 I = \begin{bmatrix}-1+i & 2 & 1\\-2 & 3+i & 2\\2 & -3 & -2+i\end{bmatrix}$$

So, the relationship $(A-\lambda_2 I)\vec{v}_2 = \vec{0}$

Leads to three possible equations: $(-1+i)x+2y+z=0$, $-2x+(3+i)y+2z=0$, and $2x-3y+(-2+i)z=0$. This one will take more manipulation. An example we could take is: first, solve for x in equations 2 and 3, then set them equal to each other.

$$x = \frac{(3+i)y+2z}{2}$$

$$x = \frac{3y-(-2+i)z}{2}$$

$3y-(-2+i)z=(3+i)y+2z$

or, $-iz=iy$, $z=-y$. Then, substitute that into equation 1.

$(-1+i)x+2y+z=(-1+i)x+y=0$ or $y=(1-i)x$. Then, we have enough to construct the eigenvector

$$\vec{v}_2=\begin{bmatrix}1\\1-i\\-1+i\end{bmatrix}$$

For $\lambda_3=-1+i$:

$$A-\lambda_3 I=\begin{bmatrix}-1-i & 2 & 1\\ -2 & 3-i & 2\\ 2 & -3 & -2-i\end{bmatrix}$$

So, the relationship $(A-\lambda_3 I)\vec{v}_3=\vec{0}$

Leads to three possible equations: $(-1-i)x+2y+z=0$, $-2x+(3-i)y+2z=0$, and $2x-3y+(-2-i)z=0$. We can, again, solve for x in equations 2 and 3 and set them equal to each other.

$$x=\frac{(3-i)y+2z}{2}$$

$$x=\frac{3y+(2+i)z}{2}$$

$(3-i)y+2z=3y+(2+i)z$, or $-iy=iz$, or $z=-y$. Substitute this relation into equation 1:

$(-1-i)x+2y+z=(-1-i)x+y=0$, or $y=(1+i)x$. That is enough to construct our eigenvector.

$$\vec{v}_3=\begin{bmatrix}1\\1+i\\-1-i\end{bmatrix}$$

Then, our general solution looks like $\vec{x}(t)=C_1e^{\lambda_1 t}\vec{v}_1+C_2e^{\lambda_2 t}\vec{v}_2+C_3e^{\lambda_3 t}\vec{v}_3$.

$$\vec{x}(t)=C_1e^{-t}\begin{bmatrix}1\\0\\1\end{bmatrix}+C_2e^{(-1-i)t}\begin{bmatrix}1\\1-i\\-1+i\end{bmatrix}+C_3e^{(-1+i)t}\begin{bmatrix}1\\1+i\\-1-i\end{bmatrix}$$

However, we need these solutions to be real (no imaginary parts). Just as before, we can write the second and third solution using the real and imaginary parts of one solution.

So, $e^{(-1-i)t}\begin{bmatrix}1\\1-i\\-1+i\end{bmatrix}=e^{-t}(\cos(t)-i\sin(t))\begin{bmatrix}1\\1-i\\-1+i\end{bmatrix}=e^{-t}\begin{bmatrix}\cos(t)-i\sin(t)\\ \cos(t)-i\sin(t)-i\cos(t)-\sin(t)\\ -\cos(t)+i\sin(t)+i\cos(t)+\sin(t)\end{bmatrix}$

Real part: $e^{-t}\begin{bmatrix}\cos(t)\\ \cos(t)-\sin(t)\\ -\cos(t)+\sin(t)\end{bmatrix}$

Imaginary part: $e^{-t}\begin{bmatrix}-\sin(t)\\ -\cos(t)-\sin(t)\\ \cos(t)+\sin(t)\end{bmatrix}$

$$\vec{x}(t)=C_1e^{-t}\begin{bmatrix}1\\0\\1\end{bmatrix}+Ae^{-t}\begin{bmatrix}\cos(t)\\ \cos(t)-\sin(t)\\ -\cos(t)+\sin(t)\end{bmatrix}+Be^{-t}\begin{bmatrix}-\sin(t)\\ -\cos(t)-\sin(t)\\ \cos(t)+\sin(t)\end{bmatrix}$$

$$=\begin{bmatrix}C_1e^{-t}+Ae^{-t}\cos(t)-Be^{-t}\sin(t)\\ Ae^{-t}(\cos(t)-\sin(t))+Be^{-t}(-\cos(t)-\sin(t))\\ C_1e^{-t}+Ae^{-t}(-\cos(t)+\sin(t))+Be^{-t}(\cos(t)+\sin(t))\end{bmatrix}$$

■

4. Second-Order Linear Differential Equations

We get a break from vectors for a bit. Chapter 4 focuses on second order linear differential equations. So, we'll discuss solutions and applications of second order ODEs.

4.1 Homogeneous Linear Second Order Differential Equations

A homogeneous, linear, second order differential equation is always of the form $a(t)x'' + b(t)x' + c(t)x = 0$, in this a, b, and c are coefficents that depend on t, and $x(t)$ is the function satisfying this differential equation.

While this form satisfies the requirements for the name, we will focus on the simpler problem where a, b, and c are all constant coefficients, $ax'' + bx' + cx = 0$.

We already solved such a problem using a linear system of differential equations.

We defined a variable, $y = x'$, and rewrote the differential equation into the system:

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = \frac{-b}{a}y - \frac{c}{a}x$$

In this section, we'll use the same general form we used for first order systems $e^{\lambda t}$ to define the solution $x(t)$ directly from the second order ODE.

When we use the form of $e^{\lambda t}$, then $x' = \lambda x$, and $x'' = \lambda^2 x$. This results in a general form:

$a\lambda^2 x + b\lambda x + cx = 0$

Which results in the same characteristic equation we determined for systems of differential equations.

$a\lambda^2 + b\lambda + c = 0$

This results in two values for λ, and two solutions to form the general solution.

A second order differential equation requires two constraints to the problem to determine a unique solution - two conditions that determine the resulting two coefficients. You will notice that these conditions are usually in the form of initial values for x and x'.

The solution is a familiar form $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

From this form, we'll see three cases.

1. Real values for λ_1 and λ_2, resulting in two exponential function solutions that may grow or decay.
   (a) Both values are positive, resulting in exponential growth as $t \to \infty$
   (b) Both values are negative, resulting in **overdamped** solution curves and exponential decay as $t \to \infty$
   (c) One value is positive and one value is negative, resulting in exponential growth as $t \to \infty$
2. Complex values for λ_1 and λ_2, resulting in two oscillating solutions that may also grow or decay.
   (a) Purely imaginary values, resulting in **undamped** solution curves - oscillating indefinitely.
   (b) Negative real part of the complex values, resulting in **underdamped** solution curves - oscillating with decreasing amplitude.
   (c) Positive real part of the complex values, resulting in solution curves that oscillate with increasing amplitude indefinitely.
3. Repeated, single value, for λ_1 and λ_2, resulting in one exponential function, and a repeated term involving a multiple of t.
   (a) Positive λ, resulting in exponential growth as $t \to \infty$.
   (b) Negative λ, resulting in **critically damped** solution curves and exponential decay as $t \to \infty$.

■ **Example 4.1** Determine the solution to $x'' - 3x' + 2x = 0$ satisfying $x(0) = 1$ and $x'(0) = 0$.
We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.
$\lambda^2 e^{\lambda t} - 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0$
Simplifying to $\lambda^2 - 3\lambda + 2 = 0$, which we can solve by factoring or using the quadratic formula.
$\lambda_1 = 1$ and $\lambda_2 = 2$, two real positive eigenvalues.
Our general solution is $x(t) = C_1 e^t + C_2 e^{2t}$.
We determine the particular solution by applying the two initial conditions.
$x(0) = C_1 + C_2 = 1$
$x'(t) = C_1 e^t + 2C_2 e^{2t}$
$x'(0) = C_1 + 2C_2 = 0$
Solve for C_1 and C_2: $C_1 = 1 - C_2$, so $1 - C_2 + 2C_2 = 0$, or $C_2 = -1$. Then, $C_1 = 2$.
$x(t) = 2e^t - e^{2t}$, and the solution will have exponential growth causing $x \to -\infty$ as $t \to \infty$.
■

■ **Example 4.2** Determine the solution to $x'' + 3x' + 2x = 0$ satisfying $x(0) = 2$ and $x'(0) = 1$.
We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.
$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0$
Simplifying to $\lambda^2 + 3\lambda + 2 = 0$, which we can solve by factoring or using the quadratic formula.
$\lambda_1 = -2$ and $\lambda_2 = -1$, two real negative eigenvalues.
Our general solution is $x(t) = C_1 e^{-2t} + C_2 e^{-t}$.
We determine the particular solution by applying the two initial conditions.
$x(0) = C_1 + C_2 = 2$
$x'(t) = -2C_1 e^{-2t} - C_2 e^{-t}$
$x'(0) = -2C_1 - C_2 = 1$
$C_2 = 2 - C_1$, so $-2C_1 - (2 - C_1) = 1$, or $-C_1 = 3$, so $C_1 = -3$ and $C_2 = 5$.
$x(t) = -3e^{-2t} + 5e^{-t}$. This solution will decay exponentially, $x \to 0$ as $t \to \infty$. This is also an example of an **overdamped** solution.
■

■ **Example 4.3** Determine the solution to $x'' + x' - 2x = 0$ satisfying $x(0) = 0$ and $x'(0) = 1$.

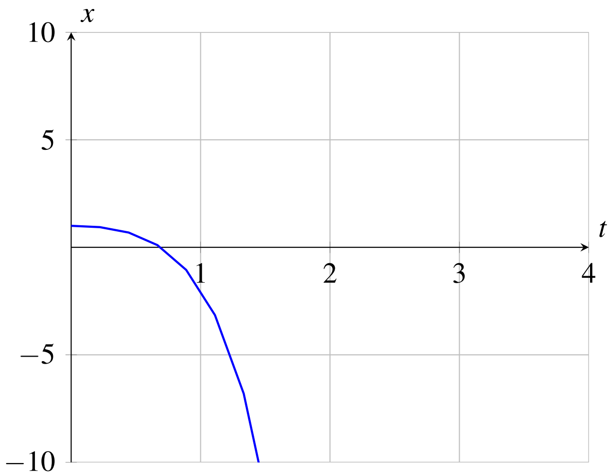


Figure 4.1: Above shows the graph of $x(t) = 2e^t - e^{2t}$, an example case with two real positive eigenvalues.

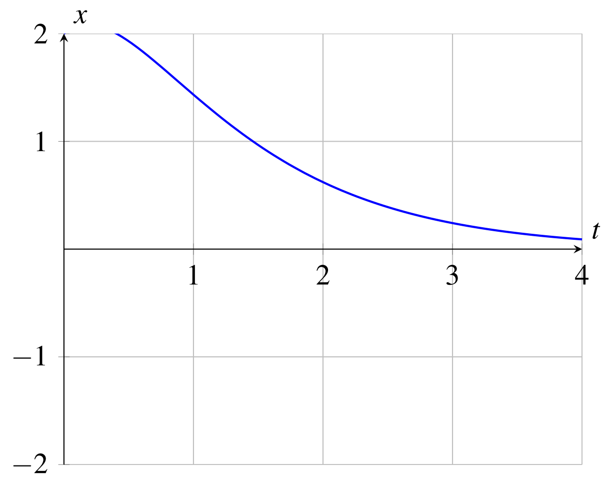


Figure 4.2: Above shows the graph of $x(t) = -3e^{-2t} + 5e^{-t}$, an example case with two real negative eigenvalues, and an example of an overdamped oscillator.

We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.
$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 2e^{\lambda t} = 0$
Simplifying to $\lambda^2 + \lambda - 2 = 0$, which we can solve by factoring or using the quadratic formula.
$\lambda_1 = -2$ and $\lambda_2 = 1$, two real eigenvalues - one negative and the other positive.
Our general solution is $x(t) = C_1 e^{-2t} + C_2 e^t$.
We determine the particular solution by applying the two initial conditions.
$x(0) = C_1 + C_2 = 0$
$x'(t) = -2C_1 e^{-2t} + C_2 e^t$
$x'(0) = -2C_1 + C_2 = 1$
$C_2 = -C_1$, so $-2C_1 + (-C_1) = 1$, or $-3C_1 = 1$, so $C_1 = -1/3$ and $C_2 = 1/3$.
$x(t) = -\dfrac{1}{3}e^{-2t} + \dfrac{1}{3}e^t$. This solution will grow exponentially, $x \to \infty$ as $t \to \infty$.

■

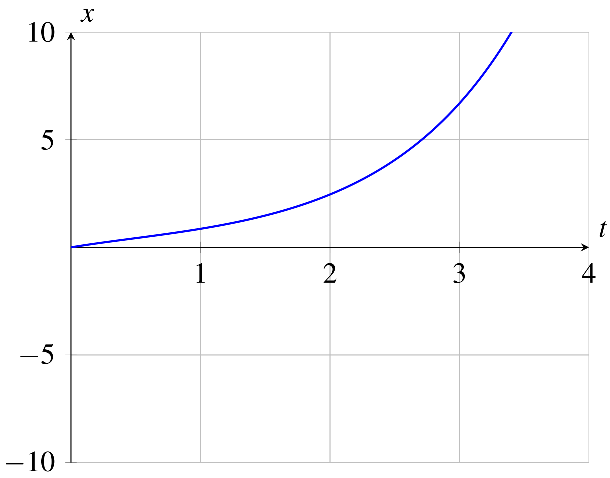


Figure 4.3: Above shows the graph of $x(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$, an example case of one positive eigenvalue and one negative.

■ **Example 4.4** Determine the solution to $x'' + 9x = 0$ satisfying $x(0) = 0.5$ and $x'(0) = 1$.

We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.

$\lambda^2 e^{\lambda t} + 9e^{\lambda t} = 0$

Simplifying to $\lambda^2 + 9 = 0$, which we can solve by factoring or using the quadratic formula.

$\lambda_1 = -3i$ and $\lambda_2 = 3i$, two purely imaginary eigenvalues.

Our general solution is $x(t) = C_1 e^{-3it} + C_2 e^{3it}$. Using Euler's formula, this becomes $x(t) = A\cos(3t) + B\sin(3t)$.

We determine the particular solution by applying the two initial conditions.

$x(0) = A + 0 = 0.5$, so $A = 0.5$.

$x'(t) = -3A\sin(3t) + 3B\cos(3t)$

$x'(0) = 3B = 1$

So, $B = 1/3$, $A = 1/2$.

$x(t) = \frac{1}{2}\cos(3t) + \frac{1}{3}\sin(3t)$. This solution will oscillate indefinitely as $t \to \infty$.

■

■ **Example 4.5** Determine the solution to $x'' + 4x' + 8x = 0$ satisfying $x(0) = 1$ and $x'(0) = 0.5$.

We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.

$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 8e^{\lambda t} = 0$

Simplifying to $\lambda^2 + 4\lambda + 8 = 0$, which we can solve by factoring or using the quadratic formula.

$\frac{-4 \pm \sqrt{16 - 4(1)(8)}}{2(1)} = \frac{-4 \pm \sqrt{-16}}{2}$

$\lambda_1 = -2 - 2i$ and $\lambda_2 = -2 + 2i$, two complex eigenvalues with a negative real part.

Our general solution is $x(t) = C_1 e^{(-2-2i)t} + C_2 e^{(-2+2i)t}$. Using Euler's formula, this becomes $x(t) = Ae^{-2t}\cos(2t) + Be^{-2t}\sin(2t)$.

We determine the particular solution by applying the two initial conditions.

$x(0) = A + 0 = 1$, so $A = 1$.

$x'(t) = -2Ae^{-2t}\cos(2t) - 2Ae^{-2t}\sin(2t) - 2Be^{-2t}\sin(2t) + 2Be^{-2t}\cos(2t)$

$x'(0) = -2A + 2B = 0.5$, or $-2 + 2B = 0.5$, so $2B = 2.5$

So, $B = 1.25$, $A = 1$.

$x(t) = e^{-2t}\cos(2t) + \frac{5}{4}e^{-2t}\sin(2t)$. This solution will oscillate and decay as $t \to \infty$.

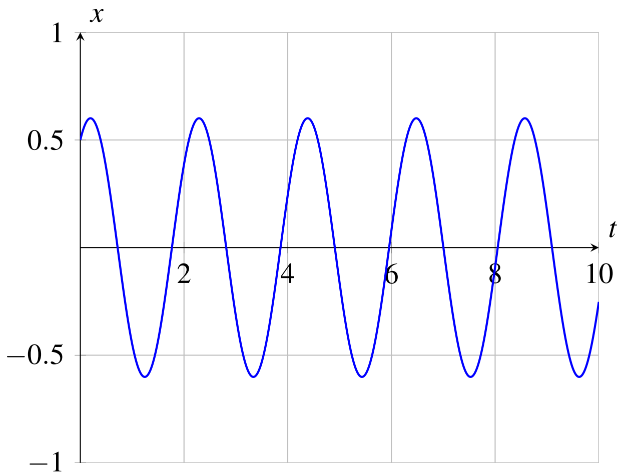


Figure 4.4: Above shows the graph of $x(t) = \frac{1}{2}\cos(3t) + \frac{1}{3}\sin(3t)$, an example case of purely imaginary exponents- oscillating indefinitely, and an example of an undamped oscillator.

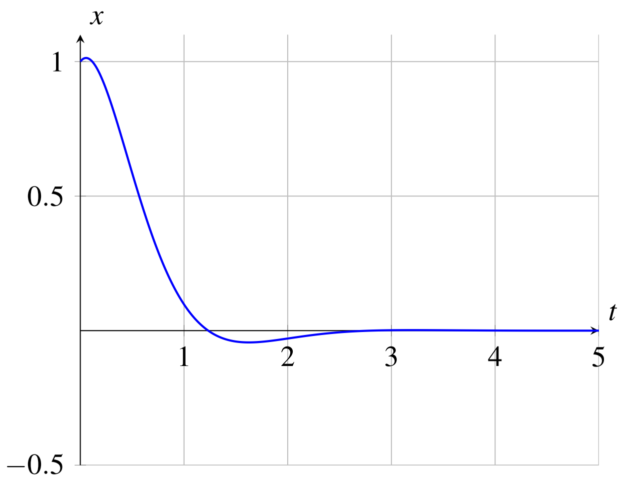


Figure 4.5: Above shows the graph of $x(t) = e^{-2t}\cos(2t) + \frac{5}{4}e^{-2t}\sin(2t)$, an example case of complex values with a negative real part - oscillating with an exponentially decaying amplitude, and an example of an underdamped oscillator.

■

■ **Example 4.6** Determine the solution to $x'' - 4x' + 8x = 0$ satisfying $x(0) = 0.2$ and $x'(0) = 0.4$.

We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.

$\lambda^2 e^{\lambda t} - 4\lambda e^{\lambda t} + 8e^{\lambda t} = 0$

Simplifying to $\lambda^2 - 4\lambda + 8 = 0$, which we can solve by factoring or using the quadratic formula.

$\frac{4 \pm \sqrt{16 - 4(1)(8)}}{2(1)} = \frac{4 \pm \sqrt{-16}}{2}$

$\lambda_1 = 2 - 2i$ and $\lambda_2 = 2 + 2i$, two complex eigenvalues with a positive real part.

Our general solution is $x(t) = C_1 e^{(2-2i)t} + C_2 e^{(22+2i)t}$. Using Euler's formula, this becomes $x(t) = Ae^{2t}\cos(2t) + Be^{2t}\sin(2t)$.

We determine the particular solution by applying the two initial conditions.

$x(0) = A + 0 = 0.2$, so $A = 0.2 = \frac{1}{5}$.
$x'(t) = 2Ae^{2t}\cos(2t) - 2Ae^{2t}\sin(2t) - 2Be^{2t}\sin(2t) + 2Be^{2t}\cos(2t)$
$x'(0) = 2A + 2B = 0.4$, or $0.4 + 2B = 0.4$, so $2B = 0$
So, $B = 0$, $A = 0.2$.
$x(t) = \frac{1}{5}e^{2t}\cos(2t)$. This solution will oscillate and the amplitude will grow as $t \to \infty$.

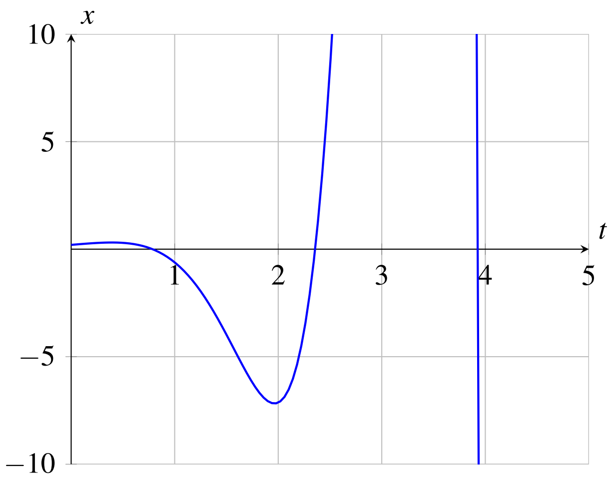


Figure 4.6: Above shows the graph of $x(t) = \frac{1}{5}e^{2t}\cos(2t)$, an example case of complex values with a positive real part - oscillating with an exponentially growing amplitude.

■

■ **Example 4.7** Determine the solution to $x'' - 2x' + x = 0$ satisfying $x(0) = 1$ and $x'(0) = 0.2$.
We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.
$\lambda^2 e^{\lambda t} - 2\lambda e^{\lambda t} + e^{\lambda t} = 0$
Simplifying to $\lambda^2 - 2\lambda + 1 = 0$, which we can solve by factoring or using the quadratic formula.
$\lambda_1 = 1$, repeated. A repeated, positive eigenvalue.
Our general solution is $x(t) = C_1e^t + C_2te^t$. (Much easier than the case using systems of equations!)
We determine the particular solution by applying the two initial conditions.
$x(0) = C_1 + 0 = 1$, so $C_1 = 1$.
$x'(t) = C_1e^t + C_2e^t + C_2te^t$
$x'(0) = C_1 + C_2 = 0.2$, or $1 + C_2 = 0.2$, so $C_2 = -0.8$
So, $C_1 = 1$, $C_2 = -0.8$.
$x(t) = e^t - 0.8te^t$. This solution will grow as $t \to \infty$.

■

■ **Example 4.8** Determine the solution to $x'' + 2x' + x = 0$ satisfying $x(0) = 0.5$ and $x'(0) = 1$.
We start by seeking a solution of the form $e^{\lambda t}$ and substituting its form into the ODE.
$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} = 0$
Simplifying to $\lambda^2 + 2\lambda + 1 = 0$, which we can solve by factoring or using the quadratic formula.
$\lambda_1 = -1$, repeated. A repeated, negative eigenvalue.
Our general solution is $x(t) = C_1e^{-t} + C_2te^{-t}$. (Much easier than the case using systems of equations!)
We determine the particular solution by applying the two initial conditions.
$x(0) = C_1 + 0 = 0.5$, so $C_1 = 0.5$.

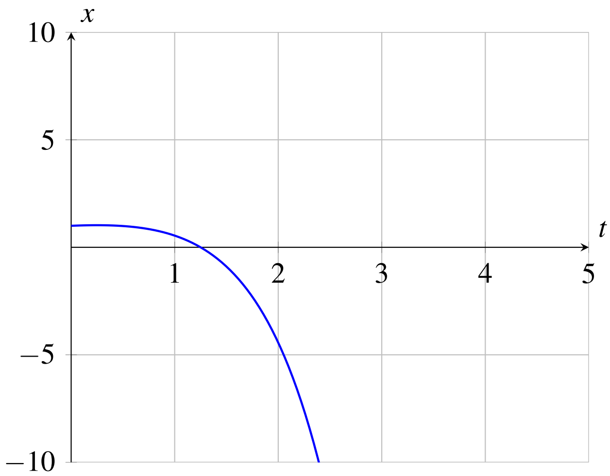


Figure 4.7: Above shows the graph of $x(t) = e^t - 0.8te^t$, an example case of a repeated positive value which will grow indefinitely toward $-\infty$ as $t \to \infty$.

$x'(t) = -C_1e^{-t} + C_2e^{-t} - C_2te^{-t}$
$x'(0) = -C_1 + C_2 = 1$, or $-0.5 + C_2 = 1$, so $C_2 = 1.5$
So, $C_1 = 0.5$, $C_2 = 1.5$.
$x(t) = 0.5e^{-t} + 1.5te^{-t}$. This solution will decay as $t \to \infty$.

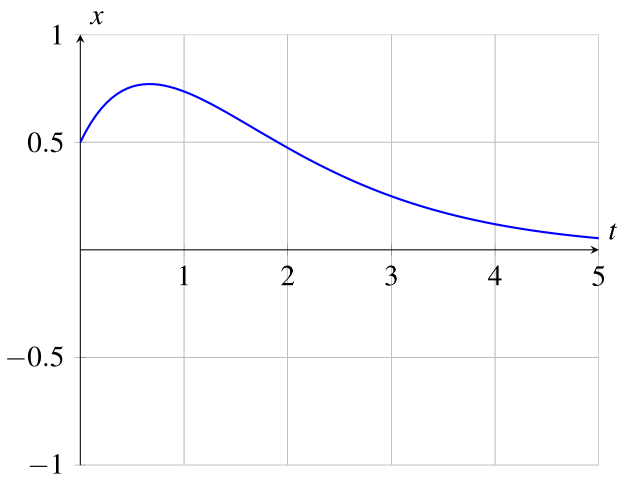


Figure 4.8: Above shows the graph of $x(t) = 0.5e^{-t} + 1.5te^{-t}$, an example case of a repeated negative value which will decay toward zero as $t \to \infty$ and an example of a critically damped oscillator.

■

4.1.1 Applications

Two primary applications of second order linear differential equations are Circuits and Oscillators (mass-spring, pendulum, etc.) We briefly discussed mass-spring systems earlier, and the form of the equations is consistent for other oscillators.

$mx'' + \gamma x' + kx = 0$

We consider four cases: (images of the graphs included above in the full set of cases)

1. Undamped, occurs when $\gamma = 0$.
2. Underdamped, occurs when $\gamma^2 < 4mk$.
3. Critically damped, occurs when $\gamma^2 = 4mk$.
4. Overdamped, occurs when $\gamma^2 > 4mk$.

■ **Example 4.9** Construct and solve the differential equation modeling the motion of a mass-spring system with a mass of $3kg$, a damping constant of $6kg/s$, and a spring constant of $2N/m$ when it is pulled $2cm$ off of equilibrium and released.

Problem: $3x'' + 6x' + 2x = 0$, $x(0) = 0.02$ and $x'(0) = 0$.

$\gamma^2 = 36$, $4mk = 24$, so this is an overdamped oscillator. It will not oscillate, but will just pull back toward equilibrium.

Seek solution of the form $e^{\lambda t}$:

$3\lambda^2 + 6\lambda + 2 = 0$

$\lambda = \dfrac{-6 \pm \sqrt{36-24}}{2*3} = \dfrac{-6 \pm 2\sqrt{3}}{6} = -1 \pm \dfrac{\sqrt{3}}{3}$

$\lambda_1 = -1 - \dfrac{\sqrt{3}}{3}$, $\lambda_2 = -1 + \dfrac{\sqrt{3}}{3}$ (both eigenvalues are negative)

$x(t) = C_1 e^{(-1-\sqrt{3}/3)t} + C_2 e^{(-1+\sqrt{3}/3)t}$

$x(0) = C_1 + C_2 = 0.02$

$x'(t) = (-1-\sqrt{3}/3)C_1 e^{(-1-\sqrt{3}/3)t} + (-1+\sqrt{3}/3)C_2 e^{(-1+\sqrt{3}/3)t}$

$x'(0) = (-1-\sqrt{3}/3)C_1 + (-1+\sqrt{3}/3)C_2 = 0$

$C_2 = \dfrac{1+\sqrt{3}/3}{-1+\sqrt{3}/3} C_1$

$C_1 \left(1 + \dfrac{1+\sqrt{3}/3}{-1+\sqrt{3}/3}\right) = 0.02$

$C_1 = 0.02 \dfrac{-1+\sqrt{3}/3}{2\sqrt{3}/3} \approx -0.00732$

$C_2 = \dfrac{1+\sqrt{3}/3}{-1+\sqrt{3}/3}(-0.007) \approx 0.0273$

$x(t) = -0.0073 e^{(-1-\sqrt{3}/3)t} + 0.0273 e^{(-1+\sqrt{3}/3)t}$

■

RLC Circuits model the current flowing through a circuit composed of a resistor, an inductor, and a capacitor. However, to get the second order differential equation, we construct the equation in terms of charge, Q.

$LQ'' + RQ' + \dfrac{Q}{C} = E(t)$

L is the inductance

R is the resistance

C is the capacitance

$E(t)$ is an external, impressed voltage on the circuit.

To solve these problems, we need to expand from homogeneous second order linear differential equations to nonhomogeneous equations.

4.2 Nonhomogeneous Second Order Differential Equations

The homogeneous equations have no external functions driving the behavior of the solution, so the solutions are formed completely by the properties of the system. When we introduce a function, we

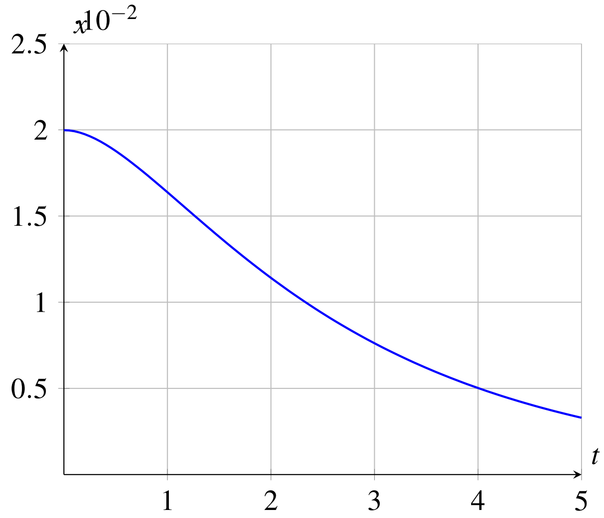


Figure 4.9: Above shows the graph of $x(t) = -0.0073e^{(-1-\sqrt{3}/3)t} + 0.0273e^{(-1+\sqrt{3}/3)t}$, an example case of an overdamped oscillator.

refer to it as a forcing function, on the right side of the equation - it will change the behavior of our solution.

$ax'' + bx' + cx = f(t)$ is a general form for a second order linear differential equation with a forcing function.

To solve these, we seek two solution forms to construct the solution. The first is the homogeneous solution, x_h, which is the solution to the homogeneous problem and any initial conditions. The second is the particular solution, x_p, which only satisfies the nonhomogeneous equation to satisfy the forcing function.

The first solution was all we did in the last section, so we will focus on how to construct the second solution.

4.2.1 Method of Undetermined Coefficients

A natural choice for determining the particular solution is to seek a solution of a similar form to the given forcing function: $f(t)$. This is the method of Undetermined Coefficients.

Type of Forcing function	Choice for Undetermined Coefficients
Constant function, $f(t) = c$	$x_p = At^2 + Bt + C$
Polynomial of degree n	Polynomial of degree $n+2$
Exponential function, $f(t) = ce^{kt}$	$x_p = Ae^{kt}$
Exponential function with same exponent as x_h	$x_p = Ate^{kt}$
Sine **or** Cosine function $f(t) = a\cos(bt) + c\sin(bt)$	$x_p = A\cos(bt) + B\sin(bt)$
Function combinations: combine the x_p terms	
Example: $f(t) = ae^{kt}\sin(bt)$	$x_p = Ae^{kt}\cos(bt) + Be^{kt}\sin(bt)$

■ **Example 4.10** Use the Method of Undetermined coefficients to solve $x'' + 3x' + 2x = t$, $x(0) = 1$, and $x'(0) = 2$.

The solution will be defined by the sum $x(t) = x_h + x_p$, so we first determine x_h as we did in the previous section.

Seek solution $e^{\lambda t}$ to the homogeneous equation $x'' + 3x' + 2x = 0$

$\lambda^2 + 3\lambda + 2 = 0$, $\lambda = -2$ and -1

$x_h(t) = C_1e^{-2t} + C_2e^{-t}$

Then, we seek $x_p = At^3 + Bt^2 + Ct + D$ to satisfy $x'' + 3x' + 2x = t$.
$x_p'(t) = 3At^2 + 2Bt + C$, $x_p''(t) = 6At + 2B$
$6At + 2B + 3(3At^2 + 2Bt + C) + 2(At^3 + Bt^2 + Ct + D) = t$
$2At^3 + (9A + 2B)t^2 + (6A + 6B + 2C)t + (2B + 3C + 2D) = t$
$2A = 0$, $9A + 2B = 0$, $6A + 6B + 2C = 1$, and $2B + 3C + 2D = 0$
So, $A = 0$, $B = 0$, $2C = 1$, and $3C + 2D = 0$.
$C = 1/2$, $D = -3/4$, and $x_p(t) = 1/2t - 3/4$.
Double-check: $x_p'(t) = 1/2$, $x_p''(t) = 0$.
$x_p'' + 3x_p' + 2x_p = 0 + 3(1/2) + 2(1/2t - 3/4) = 3/2 + t - 3/2 = t$ ✓
$x(t) = x_h(t) + x_p(t) = C_1 e^{-2t} + C_2 e^{-t} + \frac{1}{2}t - \frac{3}{4}$
Solve for C_1 and C_2 by substituting in the initial conditions.
$x(0) = C_1 + C_2 - \frac{3}{4} = 1$
$x'(t) = -2C_1 e^{-2t} - C_2 e^{-t} + \frac{1}{2}$
$x'(0) = -2C_1 - C_2 + \frac{1}{2} = 2$
$C_2 = \frac{7}{4} - C_1$
$-2C_1 - \left(\frac{7}{4} - C_1\right) + \frac{1}{2} = 2$
$C_1 = \frac{-13}{4}$, and $C_2 = 5$.
$x(t) = \frac{-13}{4} e^{-2t} + 5e^{-t} + \frac{1}{2}t - \frac{3}{4}$

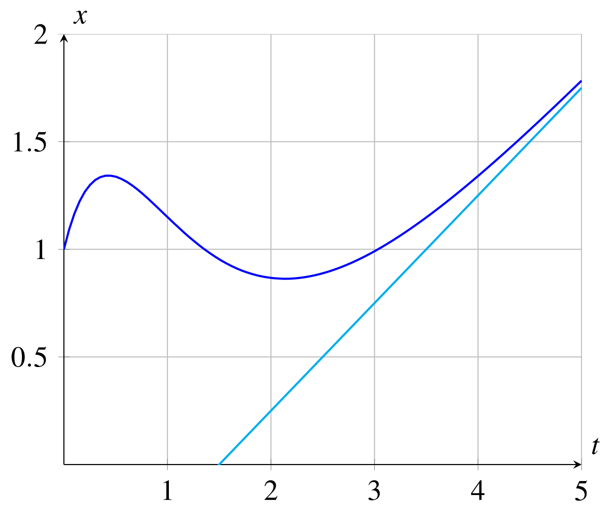


Figure 4.10: Above shows the graph of $x(t) = \frac{-13}{4} e^{-2t} + 5e^{-t} + \frac{1}{2}t - \frac{3}{4}$ in blue, an example case of an overdamped oscillator with a forcing function $f(t) = t$. The particular solution is also graphed in cyan for comparison.

In these cases, the long-term behavior of the solution will match the particular solution. In this example, the curve approaches the line $\frac{1}{2}t - \frac{3}{4}$ as $t \to \infty$ (cyan curve). ■

■ **Example 4.11** Use the Method of Undetermined coefficients to solve $x'' + 4x' + 5x = e^t$, $x(0) = 0.2$, and $x'(0) = 0.5$.

The solution will be defined by the sum $x(t) = x_h + x_p$, so we first determine x_h as we did in the previous section.

Seek solution $e^{\lambda t}$ to the homogeneous equation $x'' + 4x' + 5x = 0$

$\lambda^2 + 4\lambda + 5 = 0$, so $\lambda = \dfrac{-4 \pm 2i}{2} = -2 - i$ and $-2 + i$.

$x_h(t) = e^{-2t}\left(C_1 \cos(t) + C_2 \sin(t)\right)$

Then, we seek a solution of the form as our forcing function $f(t) = e^{2t}$, so we seek a particular solution $x_p(t) = Ae^t$.

$x_p'(t) = Ae^t$, $x_p''(t) = Ae^t$

Substituting these into the nonhomogeneous differential equation, $x'' - 4x' + 5x = e^t$, yields:

$x'' + 4x' + 5x = Ae^t + 4Ae^t + 5Ae^t = 10Ae^t$

So, $10Ae^t = e^t$, and $A = \dfrac{1}{10}$. Thus $x_p(t) = \dfrac{1}{10}e^t$

$x(t) = e^{-2t}\left(C_1 \cos(t) + C_2 \sin(t)\right) + \dfrac{1}{10}e^t$

Then, we apply the initial conditions to determine C_1 and C_2.

$x(0) = C_1 + \dfrac{1}{10} = 0.2$

$x'(t) = -2e^{-2t}\left(C_1 \cos(t) + C_2 \sin(t)\right) + e^{-2t}\left(-C_1 \sin(t) + C_2 \cos(t)\right) + \dfrac{1}{10}e^t$

$x'(0) = -2C_1 + C_2 + \dfrac{1}{10} = 0.5$

Combining the two equations, we find $C_1 = 0.1$ and $C_2 = 0.6$

So, $x(t) = e^{-2t}\left(\dfrac{1}{10}\cos(t) + \dfrac{3}{5}\sin(t)\right) + \dfrac{1}{10}e^t$.

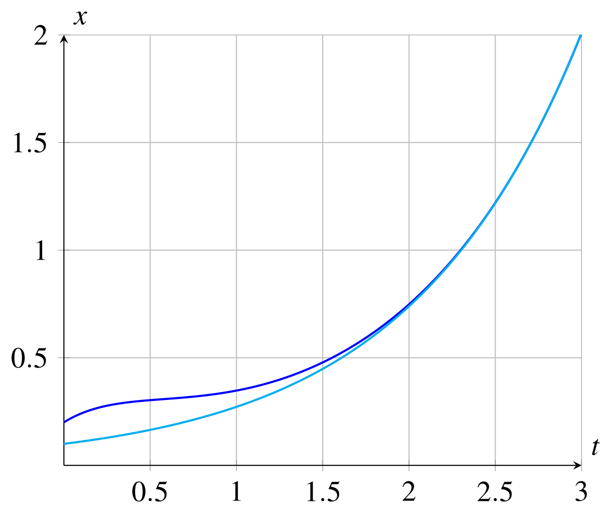


Figure 4.11: Above shows the graph of $x(t) = e^{-2t}\left(\dfrac{1}{10}\cos(t) + \dfrac{3}{5}\sin(t)\right) + \dfrac{1}{10}e^t$ in blue, an example case of an underdamped oscillator with a forcing function $f(t) = e^t$. The particular solution is graphed in cyan for comparison.

In these cases, the long-term behavior of the solution will match the particular solution. In this example, the curve approaches the curve $\dfrac{1}{10}e^t$ as $t \to \infty$ (cyan curve). ■

■ **Example 4.12** Use the Method of Undetermined coefficients to solve $x'' + 6x' + 5x = e^{2t}$, $x(0) = 0.5$, and $x'(0) = -1$.

The solution will be defined by the sum $x(t) = x_h + x_p$, so we first determine x_h as we did in the previous section.

Seek solution $e^{\lambda t}$ to the homogeneous equation $x'' + 6x' + 5x = 0$

$\lambda^2 + 6\lambda + 5 = (\lambda + 1)(\lambda + 5) = 0$, so $\lambda = -1$ and -5.

$x_h(t) = C_1 e^{-t} + C_2 e^{-5t}$

Then, we seek a solution of the form as our forcing function $f(t) = e^{2t}$, so we seek a particular solution $x_p(t) = Ae^{2t}$.

$x_p'(t) = 2Ae^{2t}$, $x_p''(t) = 4Ae^{2t}$

Substituting these into the nonhomogeneous differential equation, $x'' - 4x' + 5x = e^{2t}$, yields:

$x'' + 4x' + 5x = 4Ae^{2t} + 6 * 2Ae^{2t} + 5Ae^{2t} = 21Ae^{2t}$

So, $21Ae^{2t} = e^{2t}$, and $A = \frac{1}{21}$. Thus $x_p(t) = \frac{1}{21}e^{2t}$

$x(t) = C_1 e^{-t} + C_2 e^{-5t} + \frac{1}{21}e^{2t}$

Then, we apply the initial conditions to determine C_1 and C_2.

$x(0) = C_1 + C_2 + \frac{1}{21} = 0.5$

$x'(t) = -C_1 e^{-t} - 5C_2 e^{-5t} + \frac{2}{21}e^{2t}$

$x'(0) = -C_1 - 5C_2 + \frac{2}{21} = -1$

Combining the two equations, we find $-4C_2 + \frac{3}{21} = -0.5$, so $C_2 = 27/168 \approx 0.161$, so $C_1 = 49/168 \approx 0.292$.

So, $x(t) = \frac{49}{168}e^{-t} + \frac{27}{168}e^{-5t} + \frac{1}{21}e^{2t}$.

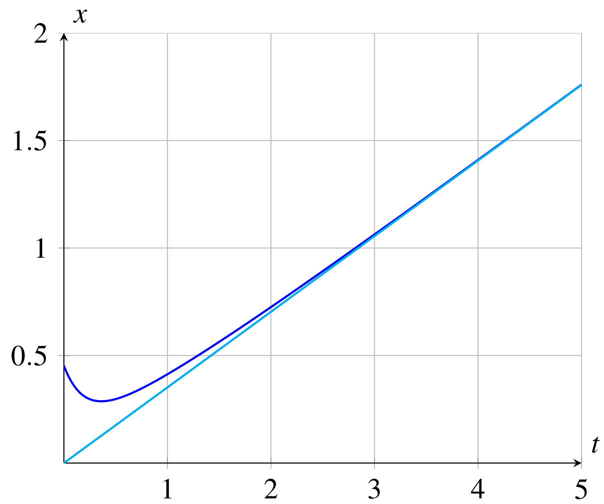


Figure 4.12: Above shows the graph of $x(t) = \frac{49}{168}e^{-t} + \frac{27}{168}e^{-5t} + \frac{1}{21}e^{2t}$ in blue, an example case of an overdamped oscillator with a forcing function $f(t) = e^{2t}$. The particular solution is graphed in cyan for comparison.

In these cases, the long-term behavior of the solution will match the particular solution. In this example, the curve approaches the curve $\frac{1}{21}e^{2t}$ as $t \to \infty$ (cyan curve).

For an extension of this example, let's consider what happens when one of the eigenvalues shows up in the forcing function.

Consider the same problem, with a forcing function $f(t) = e^{-t}$. Use the Method of Undetermined coefficients to solve $x'' + 6x' + 5x = e^{-t}$, $x(0) = 0.5$, and $x'(0) = -1$.

The homogeneous solution is the same, $x_h(t) = C_1e^{-t} + C_2e^{-5t}$. However, now that the homogeneous solution has a term of the same form as the forcing function, we have to seek an alternative solution. We seek a particular solution of the form $x_p(t) = Ate^{-t}$.

$x_p'(t) = Ae^{-t} - Ate^{-t}$, $x_p''(t) = -Ae^{-t} - Ae^{-t} + Ate^{-t}$

Substituting this into the equation: $x'' + 6x' + 5x = e^{-t}$

$-Ae^{-t} - Ae^{-t} + Ate^{-t} + 6(Ae^{-t} - Ate^{-t}) + 5Ate^{-t} = e^{-t}$

$4Ae^{-t} = e^{-t}$, so $A = \frac{1}{4}$.

$x_p(t) = \frac{1}{4}te^{-t}$

Then, we follow the same work as above to find $x(t)$.

$x(t) = C_1e^{-t} + C_2e^{-5t} + \frac{1}{4}te^{-t}$

$x(0) = C_1 + C_2 = 0.5$

$x'(t) = -C_1e^{-t} - 5C_2e^{-5t} + \frac{1}{4}e^{-t} - \frac{1}{4}te^{-t}$

$x'(0) = -C_1 - 5C_2 + \frac{1}{4} - 0 = -1$

Combining these equations, we have $-4C_2 = -.75$, $C_2 = \frac{3}{16}$, so $C_1 = \frac{5}{16}$, resulting in:

$x(t) = \frac{5}{16}e^{-t} + \frac{3}{16}e^{-5t} + \frac{1}{4}te^{-t}$

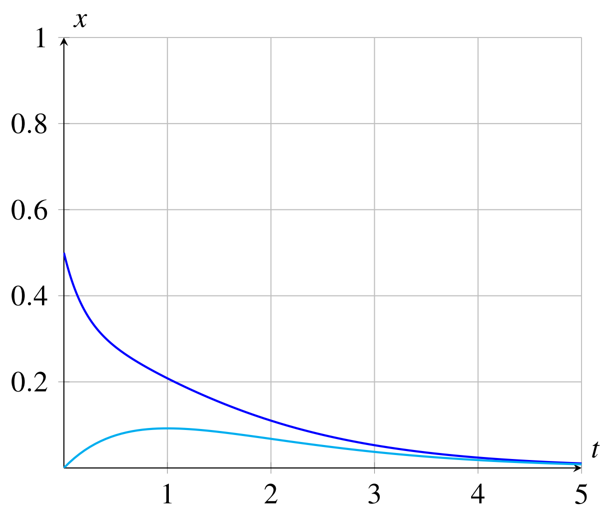


Figure 4.13: Above shows the graph of $x(t) = \frac{5}{16}e^{-t} + \frac{3}{16}e^{-5t} + \frac{1}{4}te^{-t}$ in blue, an example case of an overdamped oscillator with a forcing function $f(t) = e^{-t}$. The particular solution is graphed in cyan for comparison.

■

■ **Example 4.13** Use the Method of Undetermined coefficients to solve $x'' + 6x' + 5x = \sin(2t)$, $x(0) = 2$, and $x'(0) = 0$.

The solution will be defined by the sum $x(t) = x_h + x_p$, so we first determine x_h as we did in the previous section.

Seek solution $e^{\lambda t}$ to the homogeneous equation $x'' + 6x' + 5x = 0$

$\lambda^2+6\lambda+5=0$, so $\lambda = \dfrac{-6\pm 4}{2} = -5$ and -1.

$x_h(t) = C_1 e^{-t} + C_2 e^{-5t}$

Then, we seek a solution of the form as our forcing function $f(t) = \sin(2t)$, so we seek a particular solution $x_p(t) = A\cos(2t) + B\sin(2t)$.

$x_p'(t) = -2A\sin(2t) + 2B\cos(2t)$, $x_p''(t) = -4A\cos(2t) - 4B\sin(2t)$

Substituting these into the nonhomogeneous differential equation, $x''+6x'+5x = \sin(2t)$, yields:

$x''+6x'+5x = -4A\cos(2t) - 4B\sin(2t) + 6(-2A\sin(2t) + 2B\cos(2t)) + 5(A\cos(2t) + B\sin(2t)) = (A+12B)\cos(2t) + (-12A+B)\sin(2t) = \sin(2t)$

So, $A+12B = 0$ and $-12A+B = 1$, $A = -12B$, so $145B = 1$, $B = \dfrac{1}{145}$ and $A = \dfrac{-12}{145}$

$x_p(t) = \dfrac{-12}{145}\cos(2t) + \dfrac{1}{145}\sin(2t)$

$x(t) = C_1 e^{-t} + C_2 e^{-5t} + \dfrac{-12}{145}\cos(2t) + \dfrac{1}{145}\sin(2t)$

Then, we apply the initial conditions to determine C_1 and C_2.

$x(0) = C_1 + C_2 - \dfrac{12}{145} + 0 = 2$

$x'(t) = -C_1 e^{-t} - 5C_2 e^{-5t} + \dfrac{24}{145}\sin(2t) + \dfrac{2}{145}\cos(2t)$

$x'(0) = -C_1 - 5C_2 + \dfrac{2}{145} = 0$

Combining the two equations, we find $-4C_2 = \dfrac{300}{145}$, $C_2 = \dfrac{-75}{145}$ and $C_1 = \dfrac{377}{145}$.

So, $x(t) = \dfrac{377}{145}e^{-t} + \dfrac{-75}{145}e^{-5t} + \dfrac{1}{145}\sin(2t) - \dfrac{12}{145}\cos(2t)$.

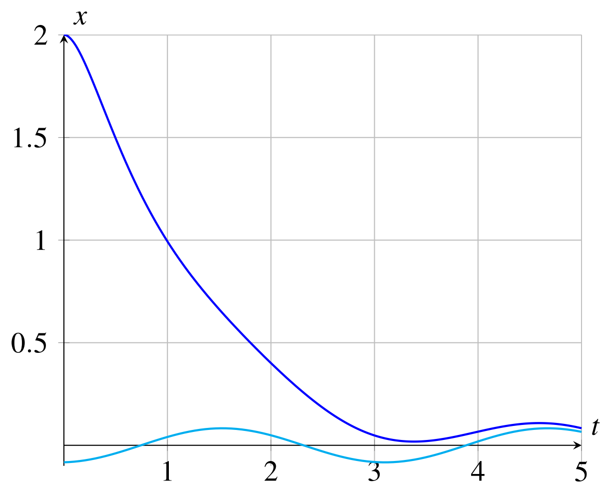


Figure 4.14: Above shows the graph of $x(t) = \dfrac{377}{145}e^{-t} + \dfrac{-75}{145}e^{-5t} + \dfrac{1}{145}\sin(2t) - \dfrac{12}{145}\cos(2t)$ in blue, an example case of an overdamped oscillator with a forcing function $f(t) = \sin(2t)$. The particular solution is graphed in cyan for comparison.

In these cases, the long-term behavior of the solution will match the particular solution. In this example, the curve approaches the curve $\dfrac{1}{145}\sin(2t) - \dfrac{12}{145}\cos(2t)$ as $t \to \infty$ (cyan curve). ■

■ **Example 4.14** Use the Method of Undetermined coefficients to solve $x''+4x'+3x = e^{-4t}\cos(2t)$, $x(0) = 1$, and $x'(0) = 2$.

The solution will be defined by the sum $x(t) = x_h + x_p$, so we first determine x_h as we did in the previous section.

Seek solution $e^{\lambda t}$ to the homogeneous equation $x'' + 4x' + 3x = 0$

$\lambda^2 + 4\lambda + 3 = 0$, so $\lambda = \dfrac{-4 \pm 2}{2} = -3$ and -1.

$x_h(t) = C_1 e^{-3t} + C_2 e^{-t}$

Then, we seek a solution of the form as our forcing function $f(t) = e^{-4t}\cos(2t)$, so we seek a particular solution $x_p(t) = Ae^{-4t}\cos(2t) + Be^{-4t}\sin(2t)$.

$x_p'(t) = -4Ae^{-4t}\cos(2t) - 2Ae^{-4t}\sin(2t) - 4Be^{-4t}\sin(2t) + 2Be^{-4t}\cos(2t) = e^{-4t}\left((-4A+2B)\cos(2t) + (-2A -\right.$

$x_p''(t) = 16Ae^{-4t}\cos(2t) + 8Ae^{-4t}\sin(2t) + 8Ae^{-4t}\sin(2t) - 4Ae^{-4t}\cos(2t) + 16Be^{-4t}\sin(2t) - 8Be^{-4t}\cos(2t) - 8Be^{-4t}\cos(2t) - 4Be^{-4t}\sin(2t)$

$= e^{-4t}\left((16A - 4A - 8B - 8B)\cos(2t) + (8A + 8A + 16B - 4B)\sin(2t)\right) = e^{-4t}\left((12A - 16B)\cos(2t) + (16A + 12B\right.$

Substituting these into the nonhomogeneous differential equation, $x'' + 4x' + 3x = e^{-4t}\cos(2t)$, yields:

$x'' + 4x' + 3x = e^{-4t}\left((12A - 16B)\cos(2t) + (16A + 12B)\sin(2t)\right) + 4e^{-4t}\left((-4A + 2B)\cos(2t) + (-2A - 4B)\sin(2\right.$ $3e^{-4t}(A\cos(2t) + B\sin(2t)) = e^{-4t}\cos(2t)$

Separating the $\cos(2t)$ terms and the $\sin(2t)$ terms, we have

$e^{-4t}\left((12A - 16B - 16A + 8B + 3A)\cos(2t) + (16A + 12B - 8A - 16B + 3B)\sin(2t)\right) = e^{-4t}\cos(2t)$

So, $-A - 8B = 1$ and $8A - B = 0$, $B = 8A$, so $-63A = 1$, $A = \dfrac{-1}{63}$ and $B = \dfrac{-8}{63}$

$x_p(t) = \dfrac{-1}{63}e^{-4t}\cos(2t) - \dfrac{8}{63}e^{-4t}\sin(2t)$

$x(t) = C_1 e^{-3t} + C_2 e^{-t} - \dfrac{1}{63}e^{-4t}\cos(2t) - \dfrac{8}{63}e^{-4t}\sin(2t)$

Then, we apply the initial conditions to determine C_1 and C_2.

$x(0) = C_1 + C_2 - \dfrac{1}{63} + 0 = 1$

$x'(t) = -3C_1 e^{-3t} - C_2 e^{-t} + \dfrac{4}{63}e^{-4t}\cos(2t) + \dfrac{2}{63}e^{-4t}\sin(2t) + \dfrac{32}{63}e^{-4t}\sin(2t) - \dfrac{16}{63}e^{-4t}\cos(2t)$

$x'(0) = -3C_1 - C_2 + \dfrac{4}{63} - \dfrac{16}{63} = 2$

Combining the two equations, we find $-2C_1 = \dfrac{202}{63}$, $C_1 = \dfrac{-101}{63}$ and $C_2 = \dfrac{165}{63}$.

So, $x(t) = \dfrac{-101}{63}e^{-3t} + \dfrac{165}{63}e^{-t} + \dfrac{-1}{63}e^{-4t}\cos(2t) - \dfrac{8}{63}e^{-4t}\sin(2t)$.

In these cases, the long-term behavior of the solution will match the particular solution. In this example, the curve approaches the curve $\dfrac{-1}{63}e^{-4t}\cos(2t) - \dfrac{8}{63}e^{-4t}\sin(2t)$ as $t \to \infty$ (cyan curve). ■

4.3 Sinusoidal Forcing

As in the last few examples, our solutions can have sinusoidal behavior in both the homogeneous and particular solutions. In this section, we will study the behavior of solutions when both solutions are sinusoidal, and use trigonometry to rewrite these equations as one function.

To begin, let's start from the undamped second order linear homogeneous differential equation for a spring-mass system:

$mx'' + kx = 0$

When we solve this system for λ, we find that our $\lambda = \sqrt{\dfrac{k}{m}}i$, and so the homogeneous solution becomes $x(t) = C_1\cos\left(\sqrt{\dfrac{k}{m}}t\right) + C_2\sin\left(\sqrt{\dfrac{k}{m}}t\right)$

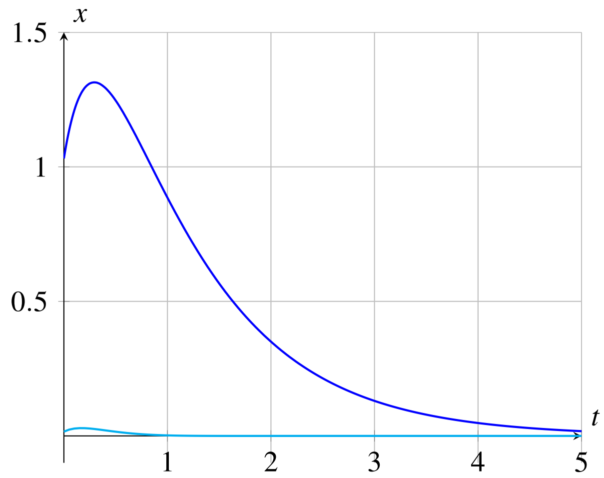


Figure 4.15: Above shows the graph of $x(t) = \frac{-101}{63}e^{-3t} + \frac{165}{63}e^{-t} + \frac{-1}{63}e^{-4t}\cos(2t) - \frac{8}{63}e^{-4t}\sin(2t)$ in blue, an example case of an overdamped oscillator with a forcing function $f(t) = e^{-4t}\cos(2t)$. The particular solution is graphed in cyan for comparison.

We refer to this as the **natural frequency** of oscillation in the system. $\omega_0 = \sqrt{\frac{k}{m}}$.

For initial conditions $x(0) = x_0$ and $x'(0) = v_0$, we can rewrite the solution as

$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$

$x(0) = C_1 = x_0$

$x'(t) = -\omega_0 C_1\sin(\omega_0 t) + \omega_0 C_2\cos(\omega_0 t)$

$x'(0) = \omega_0 C_2 = v_0$, so $C_2 = \frac{v_0}{\omega_0}$

$x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$

However, even after we've gotten this far, we don't know what the graph looks like. To sketch the graph, we convert the sum of these two functions to be written as only one function.

We seek to write a solution of the form $x(t) = A\cos(\omega_0 t - \delta)$, where A is the amplitude of oscillation formed by the sum of these two functions, ω_0 is still the frequency of oscillation, and δ is a phase shift consistent with the sum.

To convert the sum of Sine and Cosine terms to only a Cosine function, we'll use the trigonometric identity $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

We assume that the solution we have determined is in the form on the right side of the identity, and we seek the left form.

$x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) = A\cos(\omega_0 t)\cos(\delta) + A\sin(\omega_0 t)\sin(\delta)$

Then, $x_0 = A\cos(\delta)$ and $\frac{v_0}{\omega_0} = \sin(\delta)$

Recall: Polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r = \sqrt{x^2 + y^2}$, and $\theta = \arctan\left(\frac{y}{x}\right)$.

Combining this with $\cos^2(\theta) + \sin^2(\theta) = 1$, we find the following:

$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$, $\cos(\delta) = \frac{x_0}{A}$, $\sin(\delta) = \frac{v_0}{\omega_0 A}$, and thus $\delta = \arctan\left(\frac{v_0}{x_0\omega_0}\right)$.

Recall: the range of $\theta = \arctan(x)$ is only on $-\pi/2 \leq \theta \leq \pi/2$

We need to be careful about what value is appropriate for our problem.

- If $x_0 > 0$ and $\frac{v_0}{\omega_0} \geq 0$, the result will be in the first quadrant, and you can leave it as-is.
- If $x_0 > 0$ and $\frac{v_0}{\omega_0} < 0$, the result will be in the fourth quadrant and you will need to add 2π to the value from the Arctan function.
- If $x_0 < 0$ and $\frac{v_0}{\omega_0} \geq 0$, the result will be in the second quadrant and you will need to add π to the value from the Arctan function.
- If $x_0 < 0$ and $\frac{v_0}{\omega_0} < 0$, the result will be in the third quadrant and you will need to add π to the value from the Arctan function.
- If $x_0 = 0$, then the function is undefined.
  - If $\frac{v_0}{\omega_0} > 0$ then $\delta = \frac{\pi}{2}$
  - If $\frac{v_0}{\omega_0} < 0$ then $\delta = \frac{3\pi}{2}$

Recall from Trigonometry: The period of a trigonometric function is related to the frequency through $\omega_0 T = 2\pi$, or $T = \frac{2\pi}{\omega_0}$

Let's practice determining the form from a sum of two trigonometric functions.

Example 4.15 Suppose you solved an initial value problem for a homogeneous second order linear differential equation and arrived at the solution $x(t) = 2\cos(3t) + 5\sin(3t)$. Rewrite this solution as one function, $x(t) = A\cos(\omega_0 t - \delta)$.

First, determine the amplitude, A.

$A = \sqrt{2^2 + 5^2} = \sqrt{29}$

We can define ω_0 by inspection, both functions are of $3t$, so $\omega_0 = 3$.

$\delta = \arctan\left(\frac{5}{2}\right) \approx 1.19$ radians. Both coefficients are > 0 (first quadrant), so we can use this value without any adjustment.

So, $x(t) = \sqrt{29}\cos(3t - 1.19)$.

A verifying graph, for your confirmation:

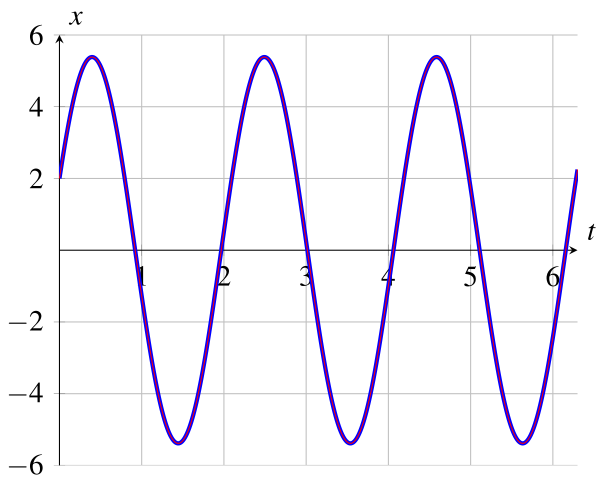


Figure 4.16: Above shows the graph of $x(t) = 2\cos(3t) + 5\sin(3t)$ in blue, and its equivalent $x(t) = \sqrt{29}\cos(3t - 1.19)$ graphed in red for comparison. The two graphs are right on top of each other, creating a cool effect of blue and red - making it appear purple - because the functions are equivalent.

■

■ **Example 4.16** Suppose you solved an initial value problem for a homogeneous second order linear differential equation and arrived at the solution $x(t) = -3\cos(\pi t) + 2\sin(\pi t)$. Rewrite this solution as one function, $x(t) = A\cos(\omega_0 t - \delta)$.

First, determine the amplitude, A.

$A = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

We can define ω_0 by inspection, both functions are of πt, so $\omega_0 = \pi$.

$\delta = \arctan\left(\dfrac{2}{-3}\right) \approx -0.588$ radians. $x_0 < 0$ and $\dfrac{v_0}{\omega_0} > 0$ (second quadrant), so we need to add π radians to the output of our arctan function. $\delta = -0.588 + \pi \approx 2.55$ radians.

So, $x(t) = \sqrt{13}\cos(\pi t - 2.55)$.

A verifying graph, for your confirmation:

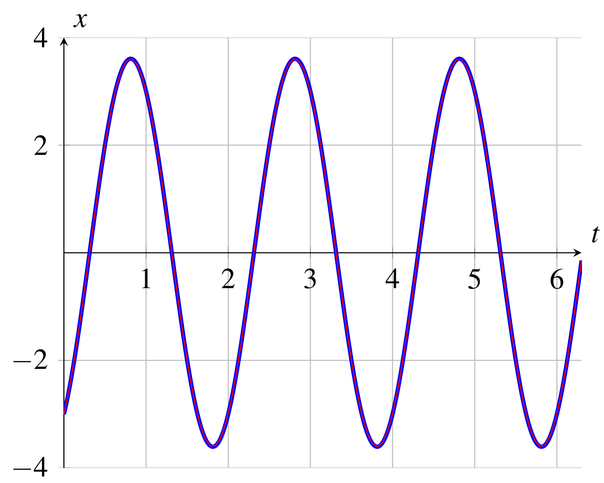


Figure 4.17: Above shows the graph of $x(t) = -3\cos(\pi t) + 2\sin(\pi t)$ in blue, and its equivalent $x(t) = \sqrt{13}\cos(\pi t - 2.55)$ graphed in red for comparison. The two graphs are right on top of each other, creating a cool effect of blue and red - making it appear purple - because the functions are equivalent.

■

Now, to build on the original problem, consider homogeneous damped mass-spring systems

$mx'' + \gamma x' + kx = 0$

In this case, we find that $\lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

There are three possible cases for the values:

1. Real values ($\gamma^2 - 4mk > 0$.) This results in an overdamped solution, and without any trigonometric functions - there is no need to rewrite the solutions.
2. Complex values ($\gamma^2 - 4mk < 0$), note that we already completed the case of purely imaginary values for the undamped cases above.) This results in an underdamped solution, and contains both exponentials and trigonometric functions. We can use it to compress the two trigonometric functions into one, as we did above.
3. Repeated values ($\gamma^2 - 4mk = 0$.) These are critically damped, and do not contain any trigonometric functions. Thus, there is no need to simplify them to rewrite the solutions.

The only case which is formed by only trigonometric functions is the Complex values case with a real part that is zero. So, we can solve for such a solution and use the trigonometric relationships to rewrite it as one trigonometric function, then sketch the solution.

■ **Example 4.17** Determine the solution to $x''+16x=0$ where $x(0)=2$ and $x'(0)=-4$. Write the solution as one trigonometric function and then sketch the graph.

First, we solve for the general solution, as usual.
Seek a solution of the form $e^{\lambda t}$:
$\lambda^2+16=0$
$\lambda=\pm 4i$
So, $x(t)=C_1\cos(4t)+C_2\sin(4t)$
Seek the solution through $x(0)=C_1+0=2$, so $C_1=2$, and $x'(t)=-4C_1\sin(4t)+4C_2\cos(4t)$.
$x'(0)=0+4C_2=-4$, so $C_2=-1$.
Thus, $x(t)=2\cos(4t)-\sin(4t)$.
To write the solution as one function we need to define the amplitude and phase shift.
$A=\sqrt{2^2+(-1)^2}=\sqrt{5}$
$\delta=\arctan(-1/2)\approx -0.4636$ radians. x_0 is positive, v_0 is negative - 4th quadrant, add 2π.
Note: this is also shown by the sign of the coefficients, the Cosine has a positive coefficient and the Sine has a negative coefficient. These are consistent because the initial values correspond to $t=0$ in these cases.
$\delta\approx 5.8195$ radians.
$x(t)=\sqrt{5}\cos(4t-5.8195)$
Sketch!
Compare:

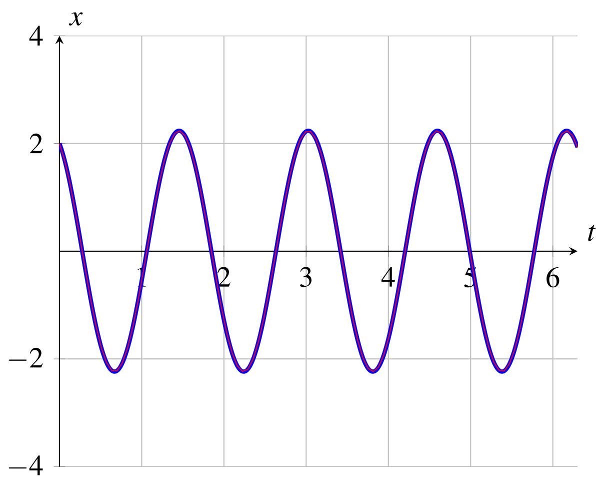


Figure 4.18: Above shows the graph of $x(t)=2\cos(4t)-\sin(4t)$ in blue, and its equivalent $x(t)=\sqrt{5}\cos(4t-5.8195)$ graphed in red for comparison. The two graphs are right on top of each other, creating a cool effect of blue and red - making it appear purple - because the functions are equivalent.

■

4.4 Forcing and Resonance

We've considered forcing functions that are sinusoidal, and the conversion of sinusoidal solutions to a single form - now we will look at what happens when we combine sinusoidal solutions with sinusoidal forcing functions.

We discussed in the last section the *natural* frequency, ω_0. Some questions we will consider in this section:

1. What happens when the frequency of a sinusoidal forcing function is different from the natural frequency? Short answer: Beats.
2. What happens when the frequency of a sinusoidal forcing function is the same as the natural frequency? Short answer: Resonance.

4.4.1 Beats

Beats occur due to the sum of two trigonometric functions with different - but close - frequencies. You may have even witnessed this behavior in your trigonometry class. For an example, look at the graph of $-\cos(6t)+\cos(5t)$:

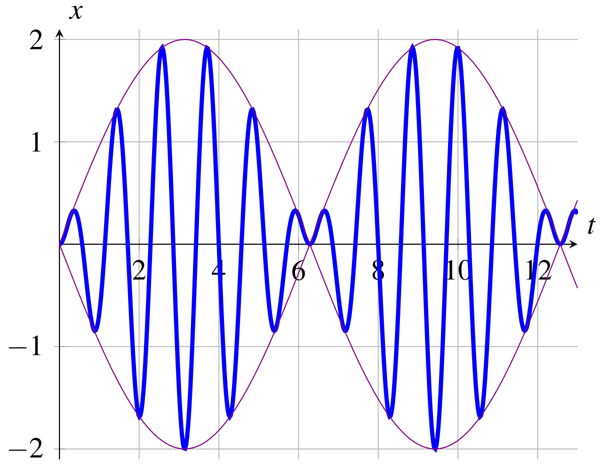


Figure 4.19: Above shows the graph of $x(t) = -\cos(6t)+\cos(5t)$ in blue. What you are seeing in the graph is the oscillation with frequency 5.5 inside an “envelope" with a frequency of 0.5. The “envelope" is plotted in violet.

In contrast, if the values are not close, there is no clear “beat." See the graph of $\cos(t)+\cos(10t)$:

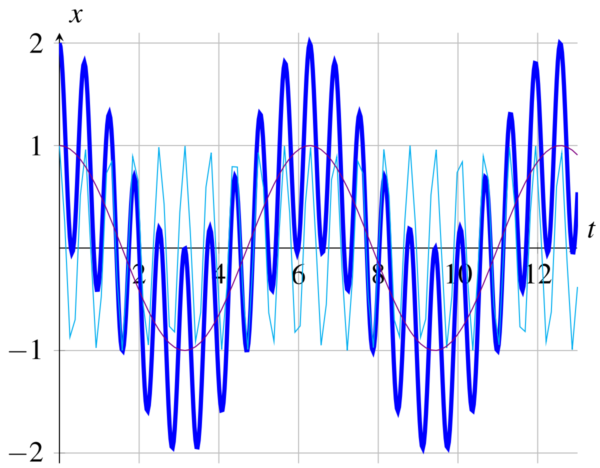


Figure 4.20: Above shows the graph of $x(t) = \cos(t)+\cos(10t)$ in blue. What you are seeing in the graph is the oscillation with the two frequencies exhibited (1 in violet and 10 in cyan), but no “beat" behavior in those oscillations.

When we combine these solutions into a single function, we use one of the trigonometric identities:

$\sin(x)+\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\sin(x)-\sin(y)=2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

In the case of our function, $x(t)=\cos(5t)-\cos(6t)$, we will use the last identity.

We define the frequencies, $\frac{x+y}{2}$ and $\frac{x-y}{2}$, as $\bar{\omega}$ and δ. $\bar{\omega}$ is the average frequency between the two functions, and δ is the difference of the two frequencies, over 2.

In our example, $\bar{\omega}=\frac{11}{2}$ and $\delta=\frac{-1}{2}$.

Since we combine the two cosine functions to $x(t)=-2\sin\left(\frac{11t}{2}\right)\sin\left(\frac{-t}{2}\right)=2\sin\left(\frac{11t}{2}\right)\sin\left(\frac{t}{2}\right)$

We use the function of the lower frequency as the "envelope", $2\sin\left(\frac{t}{2}\right)$, while the average frequency is represented in the oscillations within the envelope.

These arise in sinusoidal forcing problems as well.

■ **Example 4.18** Solve $x''+49x=3\sin(6t)$, $x(0)=0$, $x'(0)=\frac{-3}{13}$, combine into a product of functions, identify the envelope of the beat, and sketch the graph.

We solve this problem by seeking the homogeneous solution first, $x_h(t)=C_1\cos(7t)+C_2\sin(7t)$.

Then, we seek a particular solution $x_p(t)=A\cos(6t)+B\sin(6t)$, substitute it into the equation and solve for A and B.

$(-36A\cos(6t)-36B\sin(6t))+49\left(A\cos(6t)+B\sin(6t)\right)=3\sin(6t)$

$A=0$, $B=\frac{3}{13}$

$x_p(t)=\frac{3}{13}\sin(6t)$

$x(t)=C_1\cos(7t)+C_2\sin(7t)+\frac{3}{13}\sin(6t)$

Solve for C_1 and C_2, $x(0)=C_1=0$

$x'(0)=7C_2+\frac{18}{13}=\frac{-3}{13}$

$7C_2=\frac{-21}{13}$, so $C_2=\frac{-3}{13}$.

$x(t)=\frac{-3}{13}\sin(7t)+\frac{3}{13}\sin(6t)$.

We can use the second identity to simplify this.

$x(t)=\frac{3}{13}\left(\sin(6t)-\sin(7t)\right)=\frac{3}{13}\left(2\sin\left(\frac{-t}{2}\right)\cos\left(\frac{13t}{2}\right)\right)=\frac{-6}{13}\sin\left(\frac{t}{2}\right)\cos\left(\frac{13t}{2}\right)$.

In this, the envelope is defined by $\frac{-6}{13}\sin\left(\frac{t}{2}\right)$.

See the graph for comparison.

■

4.4.2 Resonance

Beats occur when the frequencies are close, but resonance occurs when the frequencies are the same. Resonance causes increase in the amplitude of oscillation that is unbounded. This is why

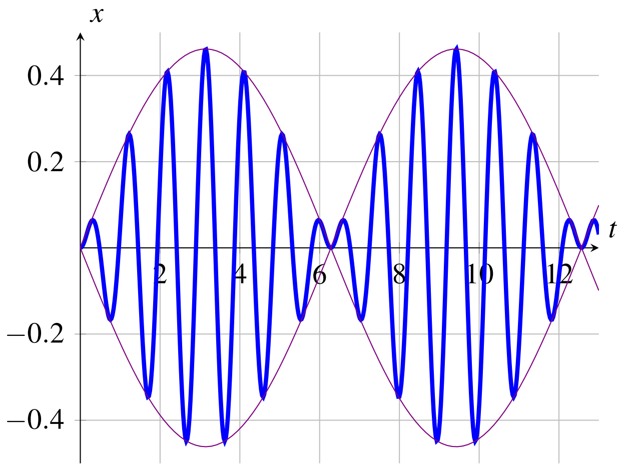


Figure 4.21: Above shows the graph of $x(t) = \frac{3}{13}(\sin(6t) - \sin(7t))$ in blue. What you are seeing in the graph is the oscillation with frequency 6.5 inside an “envelope" with a frequency of 0.5. The “envelope" is plotted in violet.

resonance is so dangerous on bridges, buildings, etc. If the frequency of the structure is matched by a force, the oscillation will have increasing amplitude that can quickly cause that bridge, building, etc. to crumble.

■ **Example 4.19** Solve $x'' + 4x = 3\sin(2t)$, $x(0) = 1$, $x'(0) = 0$, and plot the solution to visualize the resonance.

First, we determine the homogeneous solution.
$x_h(t) = C_1\cos(2t) + C_2\sin(2t)$
Then, we seek a particular solution $x_p(t) = At\cos(2t) + Bt\sin(2t)$.
$x_p''(t) = (-4A - 4Bt)\sin(2t) + (-4B + 4A)\cos(2t)$
Substituting these into $x'' + 4x = 3\sin(2t)$:
$(-4A - 4Bt)\sin(2t) + (-4B + 4A)\cos(2t) + 4(At\cos(2t) + Bt\sin(2t)) = 3\sin(2t)$
So, $-4A = 3$, $A = \frac{-3}{4}$, and $4B = 0$, $B = 0$.
$x_p(t) = \frac{-3}{4}t\cos(2t)$
$x(t) = C_1\cos(2t) + C_2\sin(2t) - \frac{3}{4}t\cos(2t)$
Then, solve for C_1 and C_2 using the initial conditions.
$x(0) = C_1 = 1$
$x'(t) = -2C_1\sin(2t) + 2C_2\cos(2t) - \frac{3}{4}\cos(2t) + \frac{6}{4}t\sin(2t)$
$x'(0) = 0 + 2C_2 - \frac{3}{4} + 0 = 0$
$C_2 = \frac{3}{8}$
$x(t) = \cos(2t) + \frac{3}{8}\sin(2t) - \frac{3}{4}t\cos(2t)$

■

4.4.3 Forced Damped Oscillators

In the real world, undamped oscillation doesn’t happen. We need to consider and analyze damped oscillators with sinusoidal forcing functions. The analysis requires the introduction of several terms

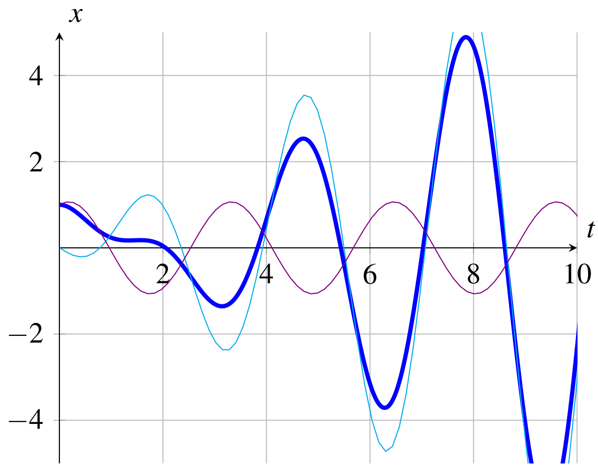


Figure 4.22: Above shows the graph of $x(t) = \cos(2t) + \frac{3}{8}\sin(2t) - \frac{3}{4}t\cos(2t)$ in blue. This is an example of resonance, as you can see in the graph - the amplitude grows as t increases. Since the particular solution results in $\frac{-3}{4}t\cos(2t)$, this is the behavior that will be exhibited as $t \to \infty$, resulting in an unbounded growth in the amplitude of oscillation. The homogeneous solution (transient solution) is included in violet, and the particular solution (steady-state solution) is in cyan.

and formulas.

Instead of focusing on Sine and Cosine, we will represent a generalized forcing function $f(t) = Ae^{i\omega t}$ in the equation $x'' + 2\delta x' + \omega_0^2 x = Ae^{i\omega t}$

This form is coming from the general equation for a harmonic oscillator $mx'' + \gamma x' + kx = F(t)$.

So, in terms of the terms from the general equation, $2c = \frac{\gamma}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

The homogeneous equation will take the form $x_h(t) = C_1 e^{(-c-\sqrt{c^2-\omega_0^2})t} + C_2 e^{(-c+\sqrt{c^2-\omega_0^2})t}$

The particular solution will take the form $x_p(t) = C_p e^{i\omega t}$ where $C_p = \frac{A}{(i\omega)^2 + 2ci\omega + \omega_0^2}$.

$x_p(t) = \frac{A}{(i\omega)^2 + 2ci\omega + \omega_0^2} e^{i\omega t}$

We define the frequency response to be the ratio of the particular solution to the forcing function:

$\frac{x_p(t)}{f(t)}$

Frequency response: $\frac{\frac{A}{(i\omega)^2 + 2ci\omega + \omega_0^2} e^{i\omega t}}{Ae^{i\omega t}} = \frac{1}{(i\omega)^2 + 2ci\omega + \omega_0^2}$

Note, there is no dependence on t in this value. It is dependent on the value of $i\omega$, so we write it as $G(i\omega) = \frac{1}{(i\omega)^2 + 2ci\omega + \omega_0^2}$.

We can rewrite this in polar form, using $G(i\omega) = |G(i\omega)|e^{-i\phi(\omega)}$.

Gain is the absolute value (or modulus) of the frequency response (Note: if the frequency response is complex, that means you multiply the value with its conjugate, then take the square root.)

Gain: $|G(i\omega)| = \sqrt{G(i\omega)\bar{G}(i\omega)} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$

From these terms we can define $x_p(t) = G(i\omega)Ae^{i\omega t} = A|G(i\omega)|e^{i(\omega t - \phi(\omega))}$, which is the general form used often in engineering and physics.

$|G(i\omega)|$ affects the amplitude and $\phi(\omega)$ affects the phase shift of the signal.

Some interesting notes about the behavior of the amplitude (relating to the other concepts in this chapter):

as $\omega \to 0$, $|G(i\omega)| \to \dfrac{1}{\omega_0^2}$

as $\omega \to \infty$, $|G(i\omega)| \to 0$

If $\omega = \omega_0$, $|G(i\omega)| \to \dfrac{1}{2\delta\omega}$

(these all come out of the form of $G(i\omega)$.)

$|G(i\omega)|$ has a relative maximum at $\omega_{max}^2 = \omega_0^2 - 2c^2$.

Resonance occurs when the amplitude of the signal is amplified by the system. The same frequency which causes a relative maximum in the steady state response is also the resonant frequency of the system.

Despite the problems caused by resonance, it's actually useful when we need to amplify a signal - particularly if you are trying to measure a known signal (seismographs, for example - weak incoming signals.)

The phase angle, as it did earlier, causes a shift in the result. $\phi(\omega) = \arccos\left(\dfrac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}\right)$

When $\omega \approx 0$, we first consider the case when $\omega = 0$, $\phi(0) = \arccos\left(\dfrac{\omega_0^2}{\omega_0^2}\right) = \arccos(1) = 0$. This implies that when $\omega \approx 0$, the phase angle is also $\phi(\omega) \approx 0$ and the result will be in phase with the forcing function $f(t)$.

For $\omega = \omega_0$, $\phi(\omega_0) = \arccos(0) = \dfrac{\pi}{2}$, and the response lags by $\phi = \pi/2$.

As $\omega \to \infty$, or $\omega >> \omega_0$, $\phi(\omega >> \omega_0) \approx \arccos\left(\dfrac{-\omega^2}{\omega^2}\right) = \arccos(-1) = \pi$. This minimizes the result because $\phi = \pi$ is in the opposite phase with the forcing function $f(t)$.

■ **Example 4.20** Determine the solution to the differential equation $x'' + 4x' + 5x = 2\cos(3t)$, $x(0) = 0$, $x'(0) = 0$, combine all terms as much as possible, and then (separately) rewrite x_p in terms of gain and phase angle. Discuss the result and compare with the graph.

First, determine the homogeneous solution by seeking a solution of the form $x_h() = e^{\lambda t}$. $\lambda = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$, so $x_h(t) = C_1 e^{-2t}\cos(t) + C_2 e^{-2t}\sin(t)$.

Then, the particular solution is a function of Sine and Cosine, which we rewrite as a general exponential $Ae^{i\omega t}$, in this case $2e^{3it}$.

$x'' + 4x' + 5x = 2e^{3it}$

$x_p(t)$ will be the real part of the solution which satisfies this, more general, equation. We seek a complex solution $x_c(t) = ae^{3it}$

Then, $-9ae^{3it} + 12iae^{3it} + 5ae^{3it} = 2e^{3it}$

$-9a + 12ia + 5a = 2$

$a = \dfrac{2}{12i - 4}$

$x_c(t) = \dfrac{2}{12i-4}e^{3it} = \dfrac{2}{12i-4}(\cos(3t) + i\sin(3t))$

To remove the i from the denominator, we must multiply by one using the conjugate.

$\dfrac{2}{12i-4} = \dfrac{2}{12i-4}\dfrac{12i+4}{12i+4} = \dfrac{24i+8}{-144-48i+48i-16} = \dfrac{-1}{160}(24i+8) = \dfrac{-1}{80}(12i+4) = \dfrac{-1}{40}(6i+2) = \dfrac{-1}{20}(3i+1)$

$x_c(t) = \frac{-1}{20}(3i+1)(\cos(3t)+i\sin(3t)) = \frac{-1}{20}(3i\cos(3t)+\cos(3t)-3\sin(3t)+i\sin(3t))$

$x_p(t) = Re[x_c(t)] = \frac{-1}{20}(\cos(3t)-3\sin(3t))$

Then, $x(t) = C_1e^{-2t}\cos(t)+C_2e^{-2t}\sin(t)+\frac{-1}{20}(\cos(3t)-3\sin(3t))$.

Applying the initial conditions, $x(0) = C_1 + \frac{-1}{20} = 0$, so $C_1 = \frac{1}{20}$.

$x'(t) = -2C_1e^{-2t}\cos(t)-C_1e^{-2t}\sin(t)-2C_2e^{-2t}\sin(t)+C_2e^{-2t}\cos(t)+\frac{-1}{20}(-3\sin(3t)-9\cos(3t))$,

$x'(0) = -2C_1+C_2+\frac{9}{20} = 0$, so $C_2 = \frac{-7}{20}$,

$x(t) = \frac{1}{20}e^{-2t}\cos(t)-\frac{7}{20}e^{-2t}\sin(t)-\frac{1}{20}(cos(3t)-3\sin(3t))$.

Using tools from the last section, we can combine the terms in our homogeneous and particular solutions to form a single term for each.

In the homogeneous solution, we'll keep e^{-2t} separate: $e^{-2t}\left(\frac{1}{20}\cos(t)-\frac{7}{20}\sin(t)\right)$, and combine the $\cos(t)$ and $\sin(t)$ terms.

$e^{-2t}\left(\sqrt{\frac{50}{400}}\cos(t-\delta)\right)$, where $\delta = \arctan(-7)+2\pi \approx 4.85$ radians. Since $C_1 > 0$ and $C_2 < 0$, we add 2π to the output.

$x_h(t) = \frac{\sqrt{2}}{4}e^{-2t}\cos(t-4.85)$.

The particular solution is combined in the same way.

$x_p(t) = \frac{-1}{20}\cos(3t)-\frac{3}{20}\sin(3t) = \sqrt{\frac{10}{400}}\cos(3t-\delta)$, where $\delta = \arctan(3)+\pi \approx 4.39$ radians. Since $C_1 < 0$ and $C_2 < 0$, we add π to the output.

$x_p(t) = \frac{\sqrt{10}}{20}\cos(3t-4.39)$.

$x(t) = \frac{\sqrt{2}}{4}e^{-2t}\cos(t-4.85)+\frac{\sqrt{10}}{20}\cos(3t-4.39)$.

To rewrite the particular solution in terms of **gain** and phase angle, we need to identify several terms:

$\omega_0 = \sqrt(5)$ from the given differential equation ($\sqrt{k/m}$.

$\omega = 3$ from the forcing term.

$c = 2$ from the given differential equation ($\gamma/2m$).

The frequency response $\frac{x_c(t)}{2e^{3it}} = \frac{1}{12i-4} = \frac{12i+4}{-160}$

From this, we determine the Gain, $|G(i\omega)| = \sqrt{\frac{12i+4}{-160}\frac{-12i+4}{-160}} = \sqrt{\frac{144+16}{160^2}} = \sqrt{\frac{1}{160}} = \frac{\sqrt{10}}{40}$.

Phase $\phi(\omega) = \arccos\left(\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4c^2\omega^2}}\right) = \arccos\left(\frac{5-9}{\sqrt{(5-9)^2+4*4*9}}\right) = \arccos\left(\frac{-4}{\sqrt{160}}\right) \approx$ 1.25 radians.

Then, $x_p(t) = |G(i\omega)|e^{-i\phi(\omega)}Ae^{i\omega t} = \frac{\sqrt{10}}{40}e^{-1.25i}2e^{3it} = \frac{\sqrt{10}}{20}e^{3it-1.25i}$.

Note: The coefficient is the same as the combined form above. The phase angle is not the same, but that is due to the exponential form. This is going to result in a product of terms, and so it doesn't translate the same.

Compare with the graph!

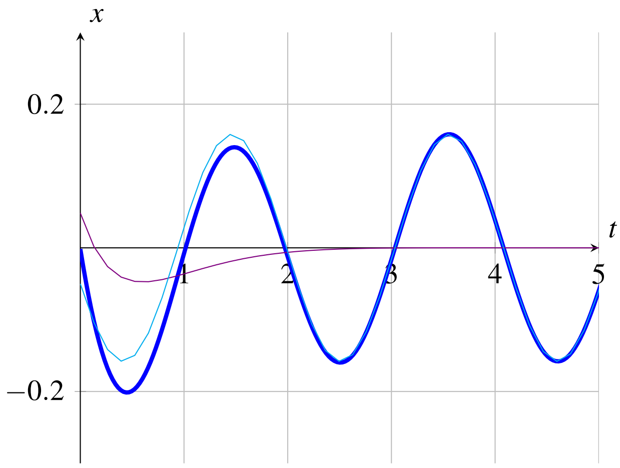


Figure 4.23: Above shows the graph of $x(t) = \frac{\sqrt{2}}{4}e^{-2t}\cos(t-4.85) + \frac{\sqrt{10}}{20}\cos(3t-4.39)$ in blue. This is an example of a damped forced harmonic oscillator problem. The homogeneous solution (transient solution) is included in violet, and the particular solution (steady-state solution) is in cyan.

■

4.5 Variation of Parameters

Variation of Parameters is an alternative approach to solving a nonhomogeneous second order linear differential equation. This technique is more general than undetermined coefficients, and will work in the cases where undetermined coefficients does not. It is also much more work to complete.

The technique stems from the solution of such problems in the form of a system of first order differential equations. Specifically, looking at a system of nonhomogeneous first order differential equations.

$\vec{x}' = P(t)\vec{x} + \vec{g}$

In the solution to this nonhomogeneous system, we seek a solution of the form $\vec{x} = u_1(t)\vec{x}_1(t) + u_2(t)\vec{x}_2(t) = \vec{x}_h(t)\cdot\vec{u}(t)$. This form contains the homogeneous solution, constants, and the particular solution.

When we apply the product rule to this form, we find $\vec{x}'(t) = u_1'(t)\vec{x}_1(t) + u_1(t)\vec{x}_1'(t) + u_2'(t)\vec{x}_2(t) + u_2(t)\vec{x}_2'(t)$

When we piece this together with the given equation using $X_h = \begin{bmatrix}\vec{x}_1 & \vec{x}_2\end{bmatrix}$, $\vec{x}'(t) = [X_h(t)\vec{u}(t)]' = X_h'\vec{u} + X_h\vec{u}' = P(t)\,[X_h\vec{u}] + \vec{g}(t)$

Notice that the homogeneous solution $\vec{x}_h' = P(t)\vec{x}_h$, so $X_h'\vec{u} = P(t)\,[X_h\vec{u}]$.

Thus, $X_h\vec{u}' = \vec{g}(t)$, and $\vec{u}' = X_h^{-1}\vec{g}(t)$. So $\vec{u}(t) = \int X_h^{-1}\vec{g}(t)dt$, and $\vec{x}_p(t) = X_h(t)\vec{u}$.

In a 2×2 system, we can define the inverse of a 2×2 matrix exactly.

For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$, $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{bmatrix}$

Now, to apply it to an example.

■ **Example 4.21** Solve the system of differential equations $\vec{x}' = \begin{bmatrix} -2 & 4 \\ -3 & 5\end{bmatrix}\vec{x} + \begin{bmatrix} 2\cos(t) \\ 3e^{-t}\end{bmatrix}$ using Variation of Parameters.

First, we need to determine the homogeneous solution: $\vec{x}_h' = \begin{bmatrix} -2 & 4 \\ -3 & 5\end{bmatrix}\vec{x}_h$

Seek the eigenvalues and eigenvectors: $\begin{bmatrix} -2-\lambda & 4 \\ -3 & 5-\lambda \end{bmatrix}$

$\det(A-\lambda I) = (-2-\lambda)(5-\lambda)+12 = 0$

$\lambda_1 = 1, \lambda_2 = 2$

Eigenvectors: $\lambda_1 = 1, A-\lambda_1 I = \begin{bmatrix} -3 & 4 \\ -3 & 4 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$

$\lambda_2 = 2, A-\lambda_2 I = \begin{bmatrix} -4 & 4 \\ -3 & 3 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}_h(t) = C_1 e^t \begin{bmatrix} 1 \\ 3/4 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1\vec{x}_1 + C_2\vec{x}_2.$

The fundamental set of solutions is given by $\vec{x}_1$ and $\vec{x}_2$.

$X_f(t) = [\vec{x}_1\vec{x}_2]$.

Before applying Variation of Parameters, we must check to see that this set is linearly independent by calculating the Wronskian.

$W[\vec{x}_1,\vec{x}_2] = \det\left(\begin{bmatrix} e^t & e^{2t} \\ 3/4e^t & e^{2t} \end{bmatrix}\right) = e^{3t} - 3/4e^{3t} = 1/4e^{3t} \neq 0$, so the solutions are linearly independent.

We then solve for $\vec{u}(t) = \int X_f^{-1} g(t)dt$, and $\vec{x}_p = X_f\vec{u}$.

$X_f = \begin{bmatrix} e^t & e^{2t} \\ 3/4e^t & e^{2t} \end{bmatrix}$, so $X_f^{-1} = \dfrac{1}{1/4e^3t}\begin{bmatrix} e^{2t} & -e^{2t} \\ -3/4e^t & e^t \end{bmatrix} = \begin{bmatrix} 4e^{-t} & -4e^{-t} \\ -3e^{-2t} & 4e^{-2t} \end{bmatrix}$

Then $\vec{u}(t) = \int \begin{bmatrix} 4e^{-t} & -4e^{-t} \\ -3e^{-2t} & 4e^{-2t} \end{bmatrix}\begin{bmatrix} 2\cos(t) \\ 3e^{-t} \end{bmatrix} dt = \int \begin{bmatrix} 8e^{-t}\cos(t) - 12e^{-2t} \\ -6e^{-2t}\cos(t) + 12e^{-3t} \end{bmatrix} dt$

We can look at each integral and then put them back together.

$\int 8e^{-t}\cos(t)dt$ needs integration by parts, and some algebraic manipulation.

u	dv
e^{-t}	$\cos(t)dt$
$-e^{-t}$	$\sin(t)$
e^{-t}	$-\cos(t)$

$\int e^{-t}\cos(t)dt = e^{-t}\sin(t) - e^{-t}\cos(t) - \int e^{-t}\cos(t)dt$

So, $2\int e^{-t}\cos(t)dt = e^{-t}\sin(t) - e^{-t}\cos(t)$, and $\int e^{-t}\cos(t)dt = \dfrac{1}{2}e^{-t}(\sin(t)-\cos(t))$

So, then $\int 8e^{-t}\cos(t) = 4e^{-t}(\sin(t)-\cos(t))$.

Then, $\int -12e^{-2t}dt = 6e^{-2t}$, so the first term $\int \left(8e^{-t}\cos(t) - 12e^{-2t}\right)dt = 4e^{-t}(\sin(t)-\cos(t)) + 6e^{-2t}$. The reason for no constant in this case is that the constant doesn't form the particular solution, it forms the homogeneous solution - which we already found. So, including it is rather arbitrary - we've already accounted for the terms from the "$+C$".

The second term, $\int -6e^{-2t}\cos(t)dt$ also needs integration by parts.

u	dv
e^{-2t}	$\cos(t)dt$
$-\dfrac{1}{2}e^{-2t}$	$\sin(t)$
$\dfrac{1}{4}e^{-2t}$	$-\cos(t)$

$\int e^{-2t}\cos(t)dt = e^{-2t}\sin(t) - \dfrac{1}{2}e^{-2t}\cos(t) - \int \dfrac{1}{4}e^{-2t}\cos(t)dt$, so $\int e^{-2t}\cos(t)dt = \dfrac{4}{5}\left(e^{-2t}\sin(t) - \dfrac{1}{2}e^{-2t}\cos(t)\right)$

$\int -6e^{-2t}\cos(t)dt = \frac{-24}{5}\left(e^{-2t}\sin(t) - \frac{1}{2}\cos(t)\right)$

$\int 12e^{-3t}dt = -4e^{-3t}$

So, $\int \left(-6e^{-2t}\cos(t) + 12e^{-3t}\right)dt = \frac{24}{10}e^{-2t}\left(2\sin(t) - \cos(t)\right) - 4e^{-3t}$.

Then $\vec{u}(t) = \begin{bmatrix} 4e^{-t}\left(\sin(t) - \cos(t)\right) + 6e^{-2t} \\ \frac{24}{10}e^{-2t}\left(2\sin(t) - \cos(t)\right) - 4e^{-3t} \end{bmatrix}$

$\vec{x}_p(t) = X_f\vec{u} = \begin{bmatrix} e^t & e^{2t} \\ 3/4e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 4e^{-t}\left(\sin(t) - \cos(t)\right) + 6e^{-2t} \\ \frac{24}{10}e^{-2t}\left(2\sin(t) - \cos(t)\right) - 4e^{-3t} \end{bmatrix} = \begin{bmatrix} 4\left(\sin(t) - \cos(t)\right) + 6e^{-t} + \frac{12}{5}\left(2\sin(t) - \text{co} \right. \\ 3\left(\sin(t) - \cos(t)\right) + \frac{9}{2}e^{-t} + \frac{12}{5}\left(2\sin(t) - \text{co} \right. \end{bmatrix}$

$\vec{x}_p(t) = \begin{bmatrix} \frac{49}{5}\sin(t) - \frac{32}{5}\cos(t) + 2e^{-t} \\ \frac{39}{5}\sin(t) - \frac{27}{5}\cos(t) + \frac{1}{2}e^{-t} \end{bmatrix}$

$\vec{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + \vec{x}_p$.

Then, $\vec{x}(t) = \begin{bmatrix} C_1e^t + C_2e^{2t} + \frac{49}{5}\sin(t) - \frac{32}{5}\cos(t) + 2e^{-t} \\ \frac{3}{4}C_1e^t + C_2e^{2t} + \frac{39}{5}\sin(t) - \frac{27}{5}\cos(t) + \frac{1}{2}e^{-t} \end{bmatrix}$ ■

Second Order ODEs can be solved in the same manner (by converting to a first order system).

■ **Example 4.22** Solve $x'' + 4x' + 4x = \cos(2t)$ using Variation of Parameters.

First, convert it to a system, $\begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(2t) \end{bmatrix}$

Then, solve the homogeneous problem.

$\lambda^2 + 4\lambda + 4 = 0$, $(\lambda + 2)^2 = 0$, so $\lambda = -2$, repeated.

$x_h(t) = C_1e^{-2t} + C_2te^{-2t}$

$x_h'(t) = -2C_1e^{-2t} + C_2e^{-2t} - 2C_2te^{-2t}$

So, $X_f(t) = \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}$

We verify the solutions are linearly independent through the Wronskian $W[\vec{x}_1, \vec{x}_2] = \det\left(\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}\right)$

$(1-2t)e^{-4t} + 2te^{-4t} = e^{-4t} \neq 0$, so yes - they are linearly independent.

$X_f^{-1} = \frac{1}{e^{-4t}}\begin{bmatrix} (1-2t)e^{-2t} & -te^{-2t} \\ 2e^{-2t} & e^{-2t} \end{bmatrix} = \begin{bmatrix} (1-2t)e^{2t} & -te^{2t} \\ 2e^{2t} & e^{2t} \end{bmatrix}$

$X_f^{-1}\vec{g} = \begin{bmatrix} (1-2t)e^{2t} & -te^{2t} \\ 2e^{2t} & e^{2t} \end{bmatrix}\begin{bmatrix} 0 \\ \cos(2t) \end{bmatrix} = \begin{bmatrix} -te^{2t}\cos(2t) \\ e^{2t}\cos(2t) \end{bmatrix}$

$\vec{u}(t) = \int X_f^{-1}\vec{g}dt = \begin{bmatrix} \int -te^{2t}\cos(2t)dt \\ \int e^{2t}\cos(2t)dt \end{bmatrix}$

Both terms requires integration by parts (the first involves it twice to account for all three terms - a triple product).

$\vec{u}(t) = \begin{bmatrix} \frac{-1}{8}e^{2t}\left(2t\cos(2t) + (2t-1)\sin(2t)\right) \\ \frac{1}{4}e^{2t}\left(\cos(2t) + \sin(2t)\right) \end{bmatrix}$

Then, $\vec{x}_p(t) = X_f\vec{u} = \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}\begin{bmatrix} \frac{-1}{8}e^{2t}\left(2t\cos(2t) + (2t-1)\sin(2t)\right) \\ \frac{1}{4}e^{2t}\left(\cos(2t) + \sin(2t)\right) \end{bmatrix} = \begin{bmatrix} \frac{-1}{8}\left(2t\cos(2\right. \\ \frac{1}{4}\left(2t\cos(2t) + (2t-1)\right. \end{bmatrix}$

$$\begin{bmatrix} \left(\frac{-t}{4}+\frac{t}{4}\right)\cos(2t)+\left(\frac{1-2t}{8}+\frac{t}{4}\right)\sin(2t) \\ \left(\frac{t}{2}+\frac{1}{4}-\frac{t}{2}\right)\cos(2t)+\left(\frac{2t-1}{4}+\frac{1}{4}-\frac{t}{2}\right)\sin(2t) \end{bmatrix}$$

$$\vec{x}_p(t) = \begin{bmatrix} \frac{1}{8}\sin(2t) \\ \frac{1}{4}\cos(2t) \end{bmatrix}$$

Then $\vec{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + \vec{x}_p$.

However, we are solving a second order ODE, so we do not have any need of the vector solution, merely the first row.

$x(t) = C_1e^{-2t} + C_2te^{-2t} + \frac{1}{8}\sin(2t)$

We can verify this with the original equation:

$x'(t) = -2C_1e^{-2t} + C_2e^{-2t} - 2C_2te^{-2t} + \frac{1}{4}\cos(2t)$

$x''(t) = 4C_1e^{-2t} - 2C_2e^{-2t} - 2C_2e^{-2t} + 4C_2te^{-2t} - \frac{1}{2}\sin(2t)$

$x''+4x'+4x = 4C_1e^{-2t} - 4C_2e^{-2t} + 4C_2te^{-2t} - \frac{1}{2}\sin(2t) + 4\left(-2C_1e^{-2t} + C_2e^{-2t} - 2C_2te^{-2t} + \frac{1}{4}\cos(2t)\right) + 4\left(C_1e^{-2t} + C_2te^{-2t} + \frac{1}{8}\sin(2t)\right) = \left((4C_1 - 8C_1 + 4C_1) + (-4C_2 + 4C_2) + (4C_2t - 8C_2t + 4C_2t)\right)e^{-2t} + \left(\frac{-1}{2} + \frac{4}{8}\right)\sin(2t) + \cos(2t) = \cos(2t)\checkmark$

This is the same solution we would have gotten using undetermined coefficients, also. Variation of Parameters provides a more general way of determining particular solutions when the form does not meet those listed in undetermined coefficients. ■

The approach using only scalar functions is derived from this more general technique that uses the vector relations.

For a second order ODE, we may seek a particular solution of the form $x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$

Then, the derivatives are in the following form:

$x_p'(t) = u_1'(t)x_1(t) + u_1(t)x_1'(t) + u_2'(t)x_2(t) + u_2(t)x_2'(t)$

$x_p''(t) = u_1''(t)x_1(t) + u_1'(t)x_1'(t) + u_1'(t)x_1'(t) + u_1(t)x_1''(t) + u_2''(t)x_2(t) + u_2'(t)x_2'(t) + u_2'(t)x_2'(t) + u_2(t)x_2''(t)$

Recall that $x_1(t)$ and $x_2(t)$ satisfy the homogeneous equation: $x'' + bx' + cx = 0$

So, $(u_1''(t)x_1(t) + u_1'(t)x_1'(t) + u_1'(t)x_1'(t) + u_1(t)x_1''(t) + u_2''(t)x_2(t) + u_2'(t)x_2'(t) + u_2'(t)x_2'(t) + u_2(t)x_2''(t)) + b\left(u_1'(t)x_1(t) + u_1(t)x_1'(t) + u_2'(t)x_2(t) + u_2(t)x_2'(t)\right) + c\left(u_1(t)x_1(t) + u_2(t)x_2(t)\right)$ contains terms that cancel.

$= (u_1(t)x_1''(t) + bu_1(t)x_1'(t) + u_1(t)x_1(t)) + (au_2(t)x_2''(t) + bu_2(t)x_2'(t) + cu_2(t)x_2(t)) + (u_1''(t)x_1(t) + 2u_1'(t)x_1'(t) + u_2''(t)x_2(t) + 2u_2'(t)x_2'(t)) + b\left(u_1'(t)x_1(t) + u_2'(t)x_2(t)\right)$, the first two terms are zero due to the homogeneous solutions.

$= (u_1''(t)x_1(t) + 2u_1'(t)x_1'(t) + u_2''(t)x_2(t) + 2u_2'(t)x_2'(t)) + b\left(u_1'(t)x_1(t) + u_2'(t)x_2(t)\right)$ This can be rewritten as: $\frac{d}{dt}\left(u_1'(t)x_1(t) + u_2'(t)x_2(t)\right) + b\left(u_1'(t)x_1(t) + u_2'(t)x_2(t)\right) + \left(u_1'(t)x_1'(t) + u_2'(t)x_2'(t)\right)$. There are two equations to be found, $u_1(t)$ and $u_2(t)$, so another condition must be introduced. The generally-accepted assumption is $u_1'(t)x_1(t) + u_2'(t)x_2(t) = 0$, which results in the first equation reducing to $u_1'(t)x_1'(t) + u_2'(t)x_2'(t) = g(t)$. This results in two formulas for $u_1(t)$ and $u_2(t)$.

$u_1(t) = \int \frac{-x_2(t)g(t)}{W[x_1,x_2]}dt$

$u_2(t) = \int \frac{x_1(t)g(t)}{W[x_1,x_2]}dt$

However, we can actually derive these from our earlier analysis using the vector notation.

Recall: $X_h = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where the first row was $x_1(t)$ and $x_2(t)$.

Then, $X_h^{-1} = \dfrac{1}{W[x_1,x_2]} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Then, the result of $X_h^{-1}\vec{g}(t)$ where $\vec{g}(t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$, was $\begin{bmatrix} \dfrac{-x_2(t)g(t)}{W[x_1,x_2]} \\ \dfrac{x_1(t)g(t)}{W[x_1,x_2]} \end{bmatrix}$

So, when we determine $u_1(t)$ and $u_2(t)$ with those equations, it's exactly the same work we did in the system.

Additionally, the assumption made is a direct result of the equation $X_h\vec{u}' = \vec{g}(t)$, the rows are:
$u_1'(t)x_1(t) + u_2'(t)x_2(t) = 0$
and $u_1'x_1'(t) + u_2'(t)x_2'(t) = g(t)$ (our two equations that result from assuming the first.)

So, for any given second order differential equation, $x'' + bx' + cx = g(t)$, we find the homogeneous solutions, $x_1(t)$ and $x_2(t)$, then solve for $x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$ where $u_1(t) = \int \dfrac{-x_2(t)g(t)}{W[x_1,x_2]}dt$ and $u_2(t) = \int \dfrac{x_1(t)g(t)}{W[x_1,x_2]}dt$. (Recall $W[x_1,x_2] = \det\left(\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}\right)$.)

■ **Example 4.23** For the same problem: Solve $x'' + 4x' + 4x = \cos(2t)$ using Variation of Parameters.

We find the homogeneous solution first. Seek solution of the form $e^{\lambda t}$:
$\lambda^2 + 4\lambda + 4 = 0$, $(\lambda+2)^2 = 0$, $\lambda = -2$, repeated.
$x_h(t) = C_1e^{-2t} + C_2te^{-2t}$
$x_1(t) = e^{-2t}$, $x_2 = te^{-2t}$
$W[x_1,x_2] = \det\left(\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}\right) = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t} \neq 0$ (Thank goodness, it's not singular!)

Then, $u_1(t) = \int \dfrac{-te^{-2t}\cos(2t)}{e^{-4t}}dt = \dfrac{-1}{8}e^{2t}\left(2t\cos(2t) + (2t-1)\sin(2t)\right)$

$u_2(t) = \int \dfrac{e^{-2t}\cos(2t)}{e^{-4t}}dt = \dfrac{1}{4}e^{2t}\left(\cos(2t) + \sin(2t)\right)$.

Finally, $x_p(t) = \dfrac{-1}{8}e^{2t}\left(2t\cos(2t) + (2t-1)\sin(2t)\right) * e^{-2t} + \dfrac{1}{4}e^{2t}\left(\cos(2t) + \sin(2t)\right) * te^{-2t} = \dfrac{-1}{8}\left(2t\cos(2t) + (2t-1)\sin(2t)\right) + \dfrac{1}{4}t\left(\cos(2t) + \sin(2t)\right)$

This reduces to $x_p(t) = \dfrac{1}{8}\sin(2t)$, just like in the earlier example.

Then, $x(t) = C_1e^{-2t} + C_2te^{-2t} + \dfrac{1}{8}\sin(2t)$. ■

■ **Example 4.24** Solve $x'' + 2x' + x = \sin(2t)$ using Variation of Parameters.

We find the homogeneous solution first. Seek solution of the form $e^{\lambda t}$:
$\lambda^2 + 2\lambda + 1 = 0$, $(\lambda+1)^2 = 0$, $\lambda = -1$, repeated.
$x_h(t) = C_1e^{-t} + C_2te^{-t}$
$x_1(t) = e^{-t}$, $x_2 = te^{-t}$
$W[x_1,x_2] = \det\left(\begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{bmatrix}\right) = e^{-2t} - te^{-2t} + te^{-2t} = e^{-2t} \neq 0$ (Thank goodness, it's not singular!)

Then, $u_1(t) = \int \dfrac{-x_2(t)f(t)}{W[x_1,x_2]}dt = \int \dfrac{-te^{-t}\sin(2t)}{e^{-2t}}dt = \dfrac{-1}{25}e^{t}\left((-10t+4)\cos(2t) + (5t+3)\sin(2t)\right)$

$u_2(t) = \int \dfrac{x_1(t)f(t)}{W[x_1,x_2]}dt = \int \dfrac{e^{-t}\sin(2t)}{e^{-2t}}dt = \dfrac{1}{5}e^{t}\left(-2\cos(2t) + \sin(2t)\right)$.

Finally, $x_p(t) = \frac{-1}{25}e^t\left((-10t+4)\cos(2t)+(5t+3)\sin(2t)\right)*e^{-t} + \frac{1}{5}e^t\left(-2\cos(2t)+\sin(2t)\right)*$ $te^{-t} = \frac{-1}{8}\left((-10t+4)\cos(2t)+(5t+3)\sin(2t)\right)+\frac{1}{5}t\left(-2\cos(2t)+\sin(2t)\right) = \left(\frac{10t}{8}-\frac{1}{2}-\frac{2t}{5}\right)\cos(2t)+$ $\left(\frac{-5t}{8}-\frac{3}{8}+\frac{t}{5}\right)$

This reduces to $x_p(t) = \left(\frac{17t}{20}-\frac{1}{2}\right)\cos(2t)+\left(\frac{-17t}{40}-\frac{3}{8}\right)\sin(2t)$.

Then, $x(t) = C_1e^{-t}+C_2te^{-t}+\left(\frac{17t}{20}-\frac{1}{2}\right)\cos(2t)+\left(\frac{-17t}{40}-\frac{3}{8}\right)\sin(2t)$. ∎

5. Nonlinear Systems

This chapter includes content for solving nonlinear systems, so it will build heavily on content from Chapters 2 and 3. The big concept we will use in nonlinear systems is: if you have a nonlinear system, linearize it. Then, you can solve for the behavior of solutions near equilibrium values. We determined equilibrium solutions of specific nonlinear systems (predator-prey and competing species) in Chapter 2, but we will discuss them more generally here.

5.1 Linearization of Nonlinear Systems

A nonlinear system is of the form:

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

when $f(x,y)$ and $g(x,y)$ are no longer linear in terms of x and y (all the systems we solved previously).

In the previous examples, all that we did focused on the behavior of solution curves in the phase plane near equilibrium values. However, in the linear case, we only had one equilibrium value and it was always at the origin $(0,0)$.

For nonlinear systems, there can be any number of equilibrium values.

Recall: Equilibria exist when **both** $f(x,y)$ and $g(x,y) = 0$. Nullclines are the paths when either $f(x,y)$ or $g(x,y) = 0$. The intersections of nullclines are the equilibrium points. x-nullclines occur when $f(x,y) = 0$ and y-nullclines occur when $g(x,y) = 0$. (Useful in the homework.)

When we solved for equilibrium values of the predator-prey models and the competing species models, there were anywhere from 2-4 equilibrium points. Now that we're dealing with general nonlinear systems, the number of equilibrium could be any nonnegative integer.

Here are the examples from Chapter 2 to refresh the work we did then:

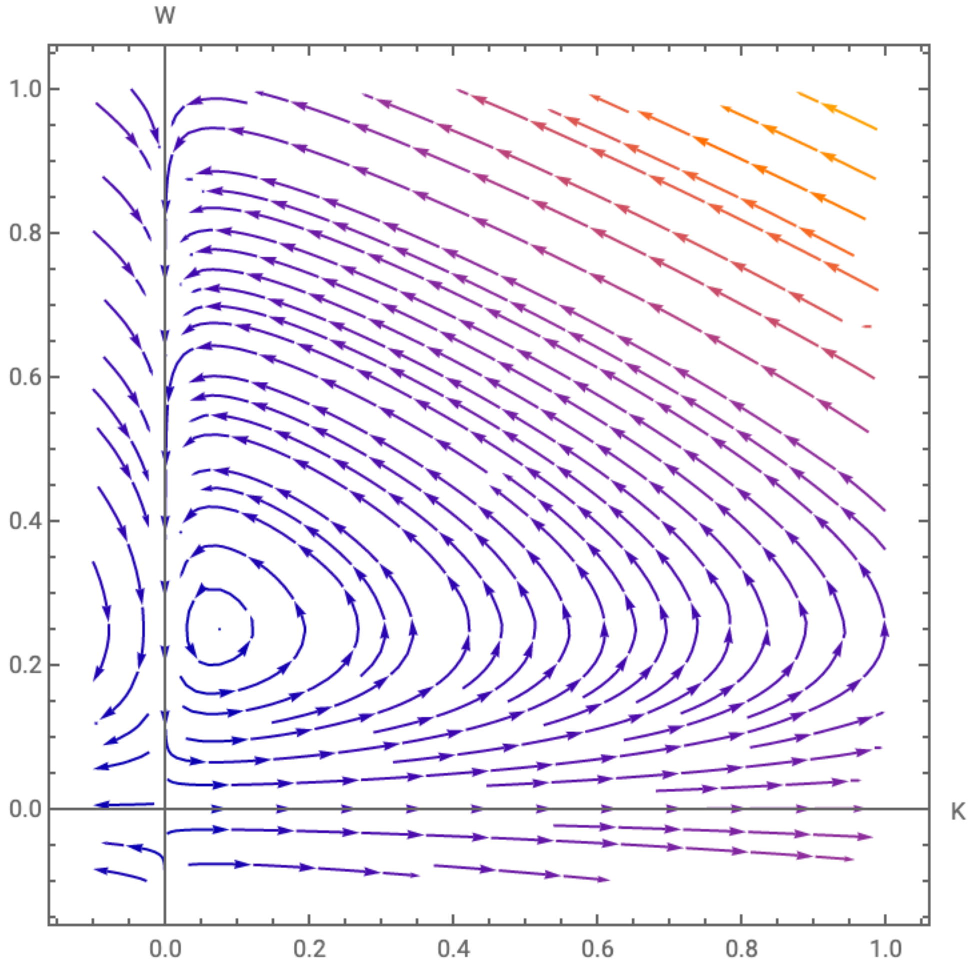
■ **Example 5.1** For the Predator-Prey system $\frac{dW}{dt} = -.2W + 3KW$ and $\frac{dK}{dt} = 2K - 8KW$, determine any equilibrium solutions, then sketch the direction field in the phase plane.

Equilibrium solutions:

$-0.2W + 3KW = 0$, if $W \neq 0$ then $K = \dfrac{1}{15}$

$2K - 8KW = 0$, if $K \neq 0$ then $W = \dfrac{1}{4}$

There is also the trivial solution, $K = 0$ and $W = 0$. If either K or $W = 0$, they both must be zero.



■

There are two equilibria, and we can see in the direction field the behavior near each equilibrium point. We will be using analysis to determine this when we don't have the direction field in front of us.

■ **Example 5.2** Analyzing the Competing Species model $\dfrac{dH}{dt} = 0.2H(1 - 0.5H - 0.25B)$, $\dfrac{dB}{dt} = 0.3B(1 - 0.4B - 0.3H)$, determine any equilibrium solutions, then sketch the direction field in the phase plane.

Equilibrium solutions exist when $\dfrac{dH}{dt}$ and $\dfrac{dB}{dt} = 0$.

$0.2H(1 - 0.5H - 0.25B) = 0$ when $H = 0$ or $H = 2 - 0.5B$

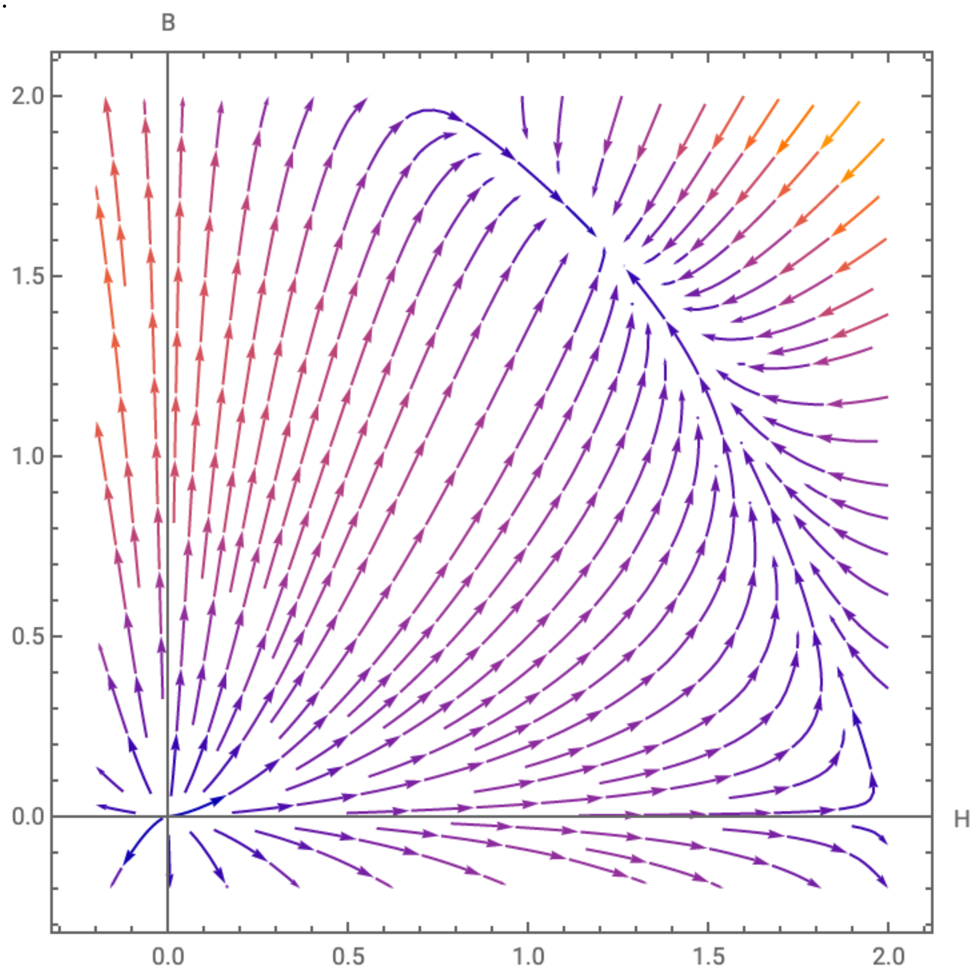
$0.3B(1 - 0.4B - 0.3H) = 0$ when $B = 0$ or $B = 2.5 - 0.75H$

So, there are equilibrium points at $(0,0)$, $(0,2.5)$, $(2,0)$ (trivial solution, and two nonphysical solutions).

The fourth point is defined by the equations $H = 2 - 0.5B$ and $B = 2.5 - 0.75H$ (the physical equilibrium point).

$B = 2.5 - 0.75 * (-0.5B + 2) = 2.5 + 0.375B - 1.5 = 1 + 0.375B$
$0.625B = 1$, $B = 1.6$
$H = -0.75B + 2.5 = -0.75 * 1.6 + 2.5 = 1.3$
Equilibrium point at $(1.3, 1.6)$.

We can sketch the direction field by selecting values of H and B, then plotting B'/H' as in the last example. The visual of the direction field is clearest through a computer, so we'll view that here.



■

In this case, we had 4 equilibrium values. Just like the predator-prey case, we were able to look at the direction field. However, we'll use analysis to define the behaviors at each equilibrium point through linearization of the system.

Let's locate the equilibria of a general example first.

■ **Example 5.3** Determine the equilibria of the nonlinear system

$$\frac{dx}{dt} = 3x^3 + 6x^2y + 2x + 4y$$
$$\frac{dy}{dt} = -2x^2 + x^2y + y - 2$$

First, we set both equations equal to zero:

The first equation, $3x^3 + 6x^2y + 2x + 4y = 0$ is going to require some factoring to determine a nontrivial solution (the trivial solution is when x and $y = 0$). We can show that it becomes $(3x^2 + 2)(2y + x) = 0$, which will only have a nonzero solution for values where $2y + x = 0$, or $x = -2y$.

The second equation, $-2x^2 + x^2y - x - 2 = 0$ is also going to require some factoring to determine a nontrivial solution (in this case, the trivial solution does not cause it to be zero - it's nonhomogeneous). We can show that it becomes $(x^2+1)(y-2) = 0$.

The second equation will only equal zero when $y = 2$, so we can use the equation from the first case $x = -2y$ to determine the only equilibrium point occurs when $x = -4$, at $(-4, 2)$.

In the next sections, we'll analyze the behavior near this point and verify it with a streamplot. ■

5.1.1 Defining Stability in Nonlinear Systems

The stability analysis in Chapter 3 focused on the behavior of solutions with respect to the origin. All solutions in the plane trended toward, away, or oscillated about the origin.

In the nonlinear problems, we can have multiple equilibrium values, and as we have shown, they aren't necessarily at the origin. This provides a need to discuss stability near the equilibria specifically.

In this context, the stability will be defined by what happens when the solution starts *close* to the equilibrium value.

If a solution curve starts close to the equilibrium, and **stays close** (does not trend toward or away), we refer to the equilibrium point as **stable**. This is the same as it was previously - but it has the caveat that the behavior will change if the initial value is not sufficiently close to the equilibrium, which is due to the behavior of other equilibrium values (which was not a problem in Chapter 3, where we always had only one equilibrium and it was at the origin.) The behavior in the phase plane will look like the stable center examples from Chapter 3, but only when near the equilbrium point.

If a solution curve starts close to the equilibrium, and it **trends toward** the equilibrium point, we refer to the equilibrium point as **asymptotically stable**. This is from the notion of a limiting case, as $t \to \infty$, $\vec{x}(t) \to$ the equilibrium point. The behavior in the phase plane will look like the asymptotically stable examples in Chapter 3 (stable spiral or node. Also referred to as a sink), but only when near the equilibrium point.

If a solution curve starts close to the equilibrium, and it **trends away** from the equilibrium point, we refer to the equilibrium point as **unstable**. This is the same as before, but now - some of those solution curves my trend away from the unstable equilibrium and toward an asymptotically stable equilibrium. The behavior in the phase plane will look like the unstable examples in Chapter 3 (unstable saddle, spiral, or node. Also referred to as a source), but only when near the equlibrium point.

There will be a dominant behavior in the phase plane, which relates to the size of the associated eigenvalue and its sign.

Autonomous Nonlinear Systems

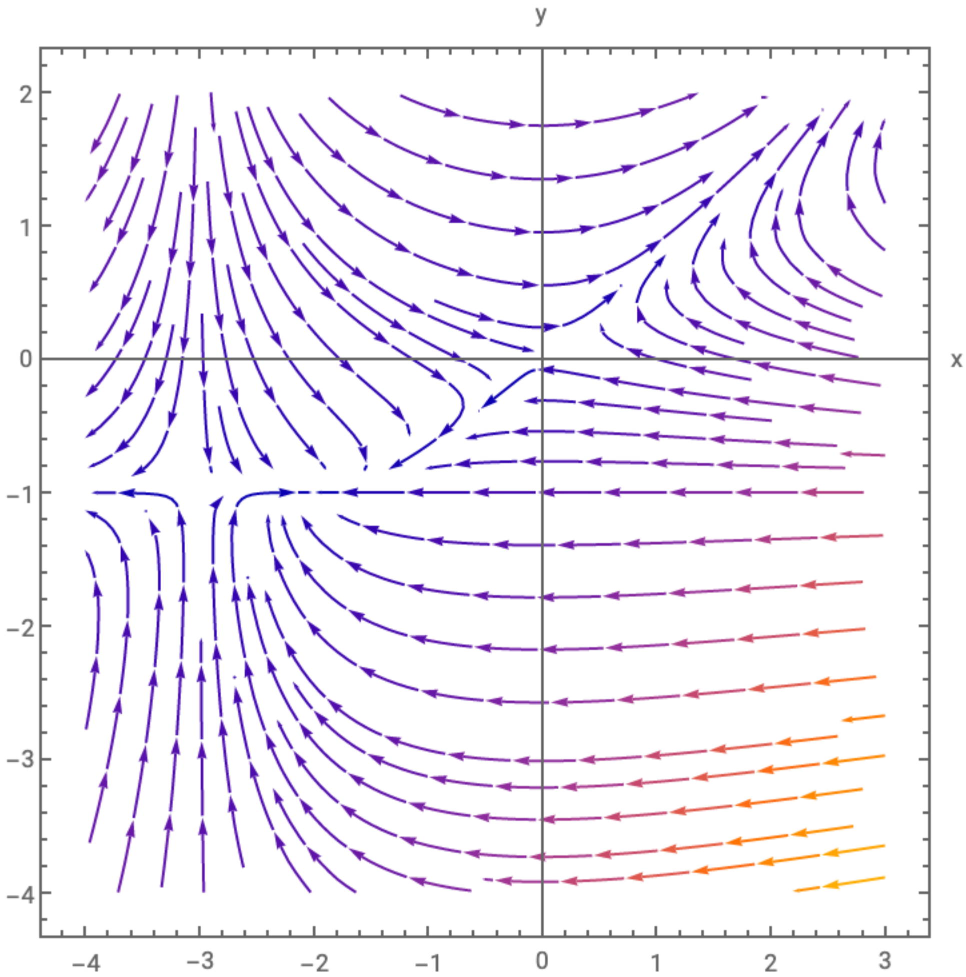
We will focus first on autonomous nonlinear systems (like the earlier examples).

If an autonomous system has at least one asymptotically stable critical point, we can define a **basin of attraction** or a **region of asymptotic stability** by the points which satisfy the requirement for asymptotic stability. The boundary of that region is called a *separatrix*. In the linear systems, the entire domain formed a region of asymptotic stability because all initial values would form a solution which approached the equlibrium point. In the nonlinear case, this will be a region we can define mathematically. However, in this course - we will merely **describe** the region of stability in words.

■ **Example 5.4** For the system $\dfrac{dx}{dt} = (3+x)(2y-x)$ and $\dfrac{dy}{dt} = x(1+y)$, determine the equilibria, classify them from the phase portrait, and then describe the region of asymptotic stability (or basin of attraction).

The first equation is $= 0$ when $x = -3$, or $x = 2y$.

The second equation is $= 0$ when $x = 0$ or $y = -1$.
These values produce 3 equilibrium points: $(0,0)$, $(-3,-1)$, and $(-2,-1)$.
We look at the StreamPlot:



Then, we can see the behaviors near each point to classify them.

$(0,0)$ has solutions approaching it along one axis, and diverging away everywhere else - it is an unstable saddle.

$(-3,-1)$ also has solutions approaching it along one axis, and diverging away everywhere else - it is an unstable saddle.

$(-2,-1)$ has all nearby solutions approaching it. It is asymptotically stable. It appears to be a node, based on the graph.

Defining the region of asymptotic stability, or paths that approach the point $(-2,-1)$: The solutions below the axis formed by the one solution approaching $(0,0)$ and to the right of the axis formed by the one solution approaching $(-2,-1)$ forms a roughly triangular shape that extends off to the right infinitely. ■

If we were to solve for the eigenvectors associated with the linearized form of this equation at each equilibrium value, we could define the separatrix by the lines associated with those eigenvectors. It's somethign we can do with the content from this course, but you do not need to complete that analysis for this course.

5.1.2 Almost Linear Systems

A linear system is of the form $\vec{x}' = A\vec{x}$, so an *almost* linear system is of the form $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$

We can test the function $\vec{g}(\vec{x})$ to determine if it is “small enough" to use the linear terms to approximate the behavior of the system.

To do this, we test the norm of the function against the solution as $\vec{x} \to 0$: $\lim_{\vec{x}\to 0} \frac{||\vec{g}(\vec{x})||}{||\vec{x}||}$

The notation $||\cdot||$ represents the 2-norm, so if $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $||\vec{x}|| = \sqrt{x^2+y^2}$. This is a case where using polar coordinates becomes convenient. $||\vec{x}|| = r$ in polar coordinates.

When we calculate $||\vec{g}(\vec{x})||$, any x can be replaced with $r\cos(\theta)$ and any y can be replaced with $r\sin(\theta)$.

■ **Example 5.5** For the system in our last example, $\frac{dx}{dt} = (3+x)(2y-x)$ and $\frac{dy}{dt} = x(1+y)$, determine if it is “almost linear."

To see this more clearly, let’s rewrite the equations into the form $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$

$\frac{dx}{dt} = -x^2 - 3x + 6y + 2xy$ and $\frac{dy}{dt} = x + xy$

So, $\vec{x}' = \begin{bmatrix} -3 & 6 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 2xy - x^2 \\ xy \end{bmatrix}$

$\vec{g}(\vec{x}) = \begin{bmatrix} 2xy - x^2 \\ xy \end{bmatrix}$

So, $||\vec{g}(\vec{x})|| = \sqrt{(2xy-x^2)^2 + (xy)^2} = \sqrt{4x^2y^2 - 4x^3y + x^4 + x^2y^2} = \sqrt{5r^4\cos^2(\theta)\sin^2(\theta) - 4r^4\cos^3(\theta)\sin(\theta) + \ldots}$

In this, we see an r^4 in every term, factor that out and use its square root. $||\vec{g}(\vec{x})|| = r^2\sqrt{5\cos^2(\theta)\sin^2(\theta) - 4\cos^3(\theta)\sin \ldots}$ we can see there is also a $\cos^2(\theta)$ in every term, factor that out and use its square root. $||\vec{g}(\vec{x})|| = r^2\cos(\theta)\sqrt{5\sin^2(\theta) - 4\cos(\theta)\sin(\theta) + \cos^2(\theta)}$. There is some additional factoring that can be done, but it’s not helpful to solving the problem. Priority is to pull any powers of r out in front of the square root, and watch out for any trigonometric functions that would be undefined (we just have Sine and Cosine, so we do not have that in this problem. Division by a trigonometric function will cause issues.)

Now, we recall that $||\vec{x}|| = r$ from earlier.

Evaluate the limit $\lim_{\vec{x}\to 0} \frac{||\vec{g}(\vec{x})||}{||\vec{x}||} = \lim_{r\to 0} \frac{r^2\cos(\theta)\sqrt{5\sin^2(\theta) - 4\cos(\theta)\sin(\theta) + \cos^2(\theta)}}{r} = \lim_{r\to 0} r\cos(\theta)\sqrt{5\sin^2(\theta) - 4\cos(\theta)\sin(\theta) + \cos^2(\theta)} = 0$ This means: yes, it’s almost linear. The norm of $\vec{g}(\vec{x})$ is smaller than the norm of $\vec{x}$, so it is acceptable to use the linearized form of the system to solve. ■

Now, to form the analysis used when solving an almost linear system.

5.1.3 Analysis

Recall: Linear Approximation of a function of multiple variables.

In Calculus III, we approximate the behavior of a function, $f(x,y)$, near a point, (x_0,y_0), by its linearization $L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$.

We will be using this concept to construct the linearization of this system of differential equations.

When we consider the system $\frac{dx}{dt} = f(x,y)$ and $\frac{dy}{dt} = g(x,y)$, we construct that linear approximation for both $f(x,y)$ and $g(x,y)$, resulting in:

$\frac{dx}{dt} \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$

$\frac{dy}{dt} \approx g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) + g(x_0,y_0)$

However, because the point (x_0, y_0) is an equilibrium point, we know that $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$ (that's how we found the equilibrium points!) Thus, the linearization for our problems becomes:

$$\frac{dx}{dt} \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$\frac{dy}{dt} \approx g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0)$$

Or, in a matrix notation:

$$\vec{x}' = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

This is a linear system of the form we solved earlier, but it has an extra bit in there with $x - x_0$ and $y - y_0$ instead of simply x and y. In order to deal with this, we apply a linear transformation to place the equilibrium point at the origin of the new coordinate system:

$u = x - x_0$ and $v = y - y_0$.

The derivatives $u' = x'$ and $v' = y'$, so all that changes is the names of our variables, the relation holds. This transformation ensures that the critical point is moved to $(0,0)$ in the (u, v) plane. Then, the approximation becomes:

$$\vec{u}' = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Now it is in the exact form of our linear system problems. So, we can solve this system as we did in Chapter 3 to define the behavior near the equilibrium point at (x_0, y_0).

The matrix given here is called the **Jacobian matrix**, J.

$$J = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}$$

As in Chapter 3, the analysis of this matrix will define the behavior of solutions near (x_0, y_0).

Additionally, assuming $\det(J) \neq 0$, we can define another way to show that a system is "almost linear."

If the functions $f(x,y)$ and $g(x,y)$ are twice differentiable, then the system $\dfrac{dx}{dt} = f(x,y)$ and $\dfrac{dy}{dt} = g(x,y)$ is almost linear

Continuing the earlier example:

Example 5.6 For the system in our last example, $\dfrac{dx}{dt} = (3+x)(2y-x)$ and $\dfrac{dy}{dt} = x(1+y)$, determine if it is "almost linear" by checking the differentiability of $f(x,y)$ and $g(x,y)$, construct the linear approximation, and determine the behavior of solutions near each equilibrium point. (Earlier we found those at $(0,0)$, $(-3,-1)$, and $(-2,-1)$.)

First, we need to expand the functions $f(x,y)$ and $g(x,y)$

$f(x,y) = -x^2 + 2xy - 3x + 6y$

$g(x,y) = x + xy$

For a function to be twice differentiable it must have a first and second derivative that exist in the real domain. (No asymptotes, cusps, etc.)

In this case, the terms are all polynomials, so they are infinitely differentiable. (No derivatives will result in asymptotes, etc.) All derivatives are defined. To show they are twice differentiable, we can determine all first and second derivatives, and evaluate their continuity.

$f_x = -2x + 2y - 3$, $f_y = 2x + 6$.

$f_{xx} = -2$, $f_{xy} = 2$, $f_{yx} = 2$, and $f_{yy} = 0$.

$g_x = 1 + y$, $g_y = x$.

$g_{xx} = 0$, $g_{xy} = 1$, $g_{yx} = 1$, $g_{yy} = 0$.

All of these are polynomials, which are continuous on the entire real domain. Yes, this system is "almost linear."

To define the linear approximation, we define the Jacobian matrix:

$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -2x+2y-3 & 2x+6 \\ 1+y & x \end{bmatrix}$

In order to determine the behavior near each equilibrium point, we will evaluate these functions at the associated equilibrium point.

1) Near $(0,0)$, $J(0,0) = \begin{bmatrix} -3 & 6 \\ 1 & 0 \end{bmatrix}$

The associated linear system to solve is $\vec{x}' = \begin{bmatrix} -3 & 6 \\ 1 & 0 \end{bmatrix} \vec{x}$ (If it is less confusing, you can replace $\vec{x}$ with $\vec{u}$.)

Then, we solve for the eigenvalues, as we did in Chapter 3.

$\det(A-\lambda I) = \det\left(\begin{bmatrix} -3-\lambda & 6 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda(3+\lambda) - 6 = \lambda^2 + 3\lambda - 6 = 0$ when $\lambda = \dfrac{-3 \pm \sqrt{9-4*1*(-6)}}{2} = \dfrac{-3 \pm \sqrt{33}}{2}$. These eigenvalues are both real, one is negative, and the other is positive ($\sqrt{33}$ is larger than 3, resulting in a positive value).

The eigenvalues tell us that the equilibrium value at $(0,0)$ is an unstable saddle. The eigenvectors tell us the vectors along which those eigenvalue trajectories propagate.

The eigenvectors are necessary to sketch the behavior, but not necessary to define the behavior. So, we will skip that for this example (a sketch is not requested.)

2) Near $(-3,-1)$, $J(-3,-1) = \begin{bmatrix} 6-2-3 & -6+6 \\ 1-1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

The associated linear system to solve is $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \vec{x}$ (If it is less confusing, you can replace $\vec{x}$ with $\vec{u}$.)

We solve for the eigenvalues (but we can see them by inspection because the off-diagonal terms are zero).

$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix}\right) = (-3-\lambda)(1-\lambda) = 0$, $\lambda = -3$ and 1. Again we have two real eigenvalues, one is negative and one is positive. The equilibrium point at $(-3,-1)$ is also an unstable saddle.

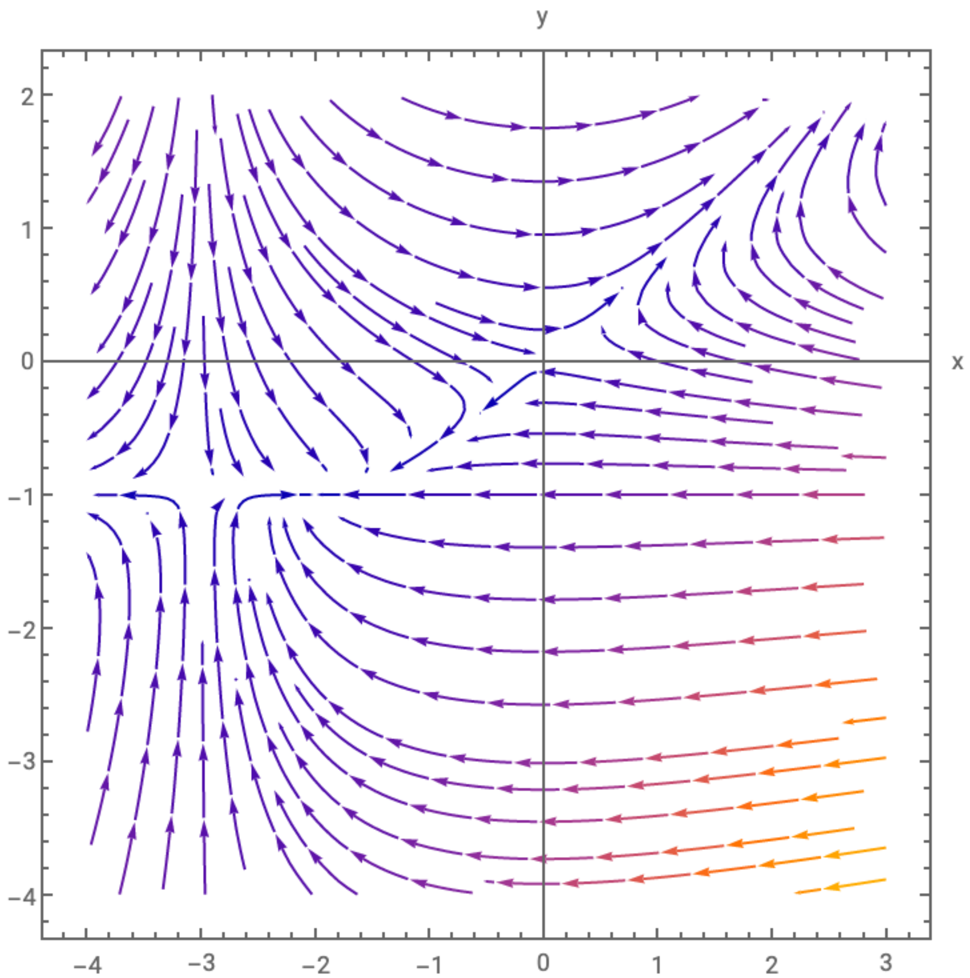
3) Near $(-2,-1)$, $J(-2,-1) = \begin{bmatrix} 4-2-3 & -4+6 \\ 1-1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}$

The associated linear system to solve is $\vec{x}' = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix} \vec{x}$ (If it is less confusing, you can replace $\vec{x}$ with $\vec{u}$.)

We solve for the eigenvalues (but, again, can see them by inspection).

$\det(A-\lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 2 \\ 0 & -2-\lambda \end{bmatrix}\right) = (-1-\lambda)(-2-\lambda) - 0 = 0$, $\lambda = -2$ and -1. In this case, both eigenvalues are real and negative. The equilibrium point at $(-2,-1)$ is an asymptotically stable node (sink).

It is not required, but we can confirm these from the StreamPlot:



■

■ **Example 5.7** For the following nonlinear system: determine all equilibrium points, then show if the system is "almost linear". If it is, define the behavior near each equilibrium point, and for any asymptotically stable equilibria - describe the basin of attraction (region of asymptotic stability).

$$\vec{x}' = \begin{bmatrix} 2x-y \\ y-x^2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ -x^2 \end{bmatrix}$$

Equilibria occur when $\vec{x}' = \vec{0}$. For these functions, that occrus when $y = 2x$ and $y = x^2$, or $2x = x^2$. This is only true for $x = 0$ and $x = 2$. If $x = 0$, $y = 0$. If $x = 2$, $y = 4$. So, the two equilibria occur at $(0,0)$ and $(2,4)$.

We can show that the system is "almost linear" through the limit, $\lim_{\vec{x}\to\vec{0}} \dfrac{||\vec{g}(\vec{x})||}{||\vec{x}||}$ or showing it is twice differentiable.

1) $\lim_{\vec{x}\to\vec{0}} \dfrac{||\vec{g}(\vec{x})||}{||\vec{x}||} = \lim_{\vec{x}\to\vec{0}} \dfrac{\sqrt{0^2+(x^2)^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \dfrac{r^2\cos^2(\theta)}{r} = 0$. So, the system is "almost linear."

OR

2) $\vec{f}_x(\vec{x}) = \begin{bmatrix} 2 \\ -2x \end{bmatrix}$, $\vec{f}_y(\vec{x}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{f}_{xx}(\vec{x}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $\vec{f}_{xy}(\vec{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{f}_{yx}(\vec{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{f}_{yy}(\vec{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
All of these are defined in the real domain, $\vec{f}(\vec{x})$ is twice differentiable.

To define the behavior near each equilibrium point, we define the Jacobian matrix: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2x & 1 \end{bmatrix}$.

1) Near $(0,0)$: $J(0,0) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

The associated linear system is $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}$.

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (2-\lambda)(1-\lambda) = 0$, $\lambda = 1, 2$. Both eigenvalues are real and positive, so the behavior near $(0,0)$ is an unstable node (source).

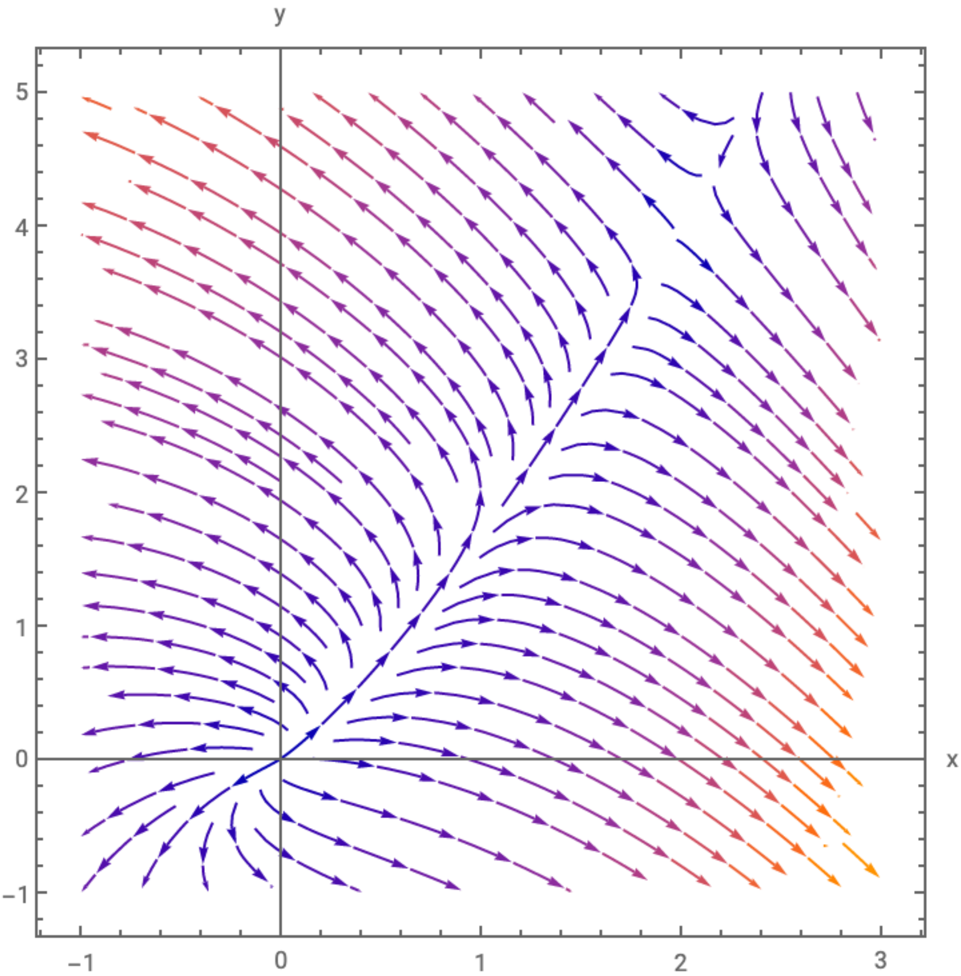
2) Near $(2,4)$: $J(2,4) = \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

The associated linear system is $\vec{x}' = \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} \vec{x}$.

$\det(A-\lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -1 \\ -4 & 1-\lambda \end{bmatrix}\right) = (2-\lambda)(1-\lambda) - 4 = \lambda^2 - 3\lambda - 2 = 0$, $\lambda = \dfrac{3 \pm \sqrt{9-4*1*(-2)}}{2} = \dfrac{3 \pm \sqrt{17}}{2}$. The eigenvalues are real, one is negative and the other is positive. The behavior near $(2,4)$ is an unstable saddle.

There is no region of asymptotic stability (basin of attraction), neither point is asymptotically stable.

View the StreamPlot to confirm:



■

■ **Example 5.8** For the following nonlinear system: determine all equilibrium points, then show if the system is “almost linear". If it is, define the behavior near each equilibrium point, and for any asymptotically stable equilibria - describe the basin of attraction (region of asymptotic stability).

$\frac{dx}{dt} = (3-x)(y+x)$ and $\frac{dy}{dt} = (1+x)(x-y)$

The first equation is zero only if $x = 3$ or $y = -x$, the second equation is zero only if $x = -1$ or $y = x$.

So, if $x = 3$, then $y = x$ means $y = 3$ to satisfy both equations, $(3,3)$.

If $x = -1$, then $y = -x$ means $y = 1$ to satisfy both equations, $(-1,1)$.

Otherwise, when is $x = -x$? Only if $x = 0$, then $y = 0$. $(0,0)$

The three equilibria are $(-1,1)$, $(0,0)$, and $(3,3)$.

To determine if the system is "almost linear," let us first expand the functions.

$f(x,y) = -x^2 - xy + 3x + 3y$, $g(x,y) = x^2 - xy + x - y$.

These are both polynomials, all derivatives exist in the real domain.

$f_x = -2x - y + 3$, $f_y = -x + 3$, $g_x = 2x - y + 1$, $g_y = -x - 1$.

$f_{xx} = -2$, $f_{xy} = -1$, $f_{yx} = -1$, $f_{yy} = 0$, $g_{xx} = 2$, $g_{xy} = -1$, $g_{yx} = -1$, $g_{yy} = 0$. (no asymptotes, etc.)

This system is "almost linear."

To define the associated linear systems, we need the Jacobian matrix.

$J = \begin{bmatrix} -2x-y+3 & -x+3 \\ 2x-y+1 & -x-1 \end{bmatrix}$

1) Near $(-1,1)$: $J(-1,1) = \begin{bmatrix} 2-1+3 & 1+3 \\ -2-1+1 & 1-1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -2 & 0 \end{bmatrix}$

The associated linear system is $\vec{x}' = \begin{bmatrix} 4 & 4 \\ -2 & 0 \end{bmatrix} \vec{x}$

Determine the eigenvalues for the associated linear system:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 4 \\ -2 & -\lambda \end{bmatrix}\right) = -\lambda(4-\lambda) + 8 = \lambda^2 - 4\lambda + 8 = 0$, $\lambda = \frac{4 \pm \sqrt{16 - 4*1*8}}{2} = \frac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i$. The eigenvalues are complex with a positive real part. The behavior near $(-1,1)$ is an unstable spiral (source).

2) Near $(0,0)$: $J(0,0) = \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix}$

The associated linear system is $\vec{x}' = \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} \vec{x}$.

We determine the eigenvalues for the associated linear system:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 3 \\ 1 & -1-\lambda \end{bmatrix}\right) = (3-\lambda)(-1-\lambda) - 3 = \lambda^2 - 2\lambda - 6 = 0$, $\lambda = \frac{2 \pm \sqrt{4 - 4*1*(-6)}}{2} = \frac{2 \pm \sqrt{28}}{2}$. The two eigenvalues are real, one is positive and the other is negative. The behavior near $(0,0)$ is an unstable saddle.

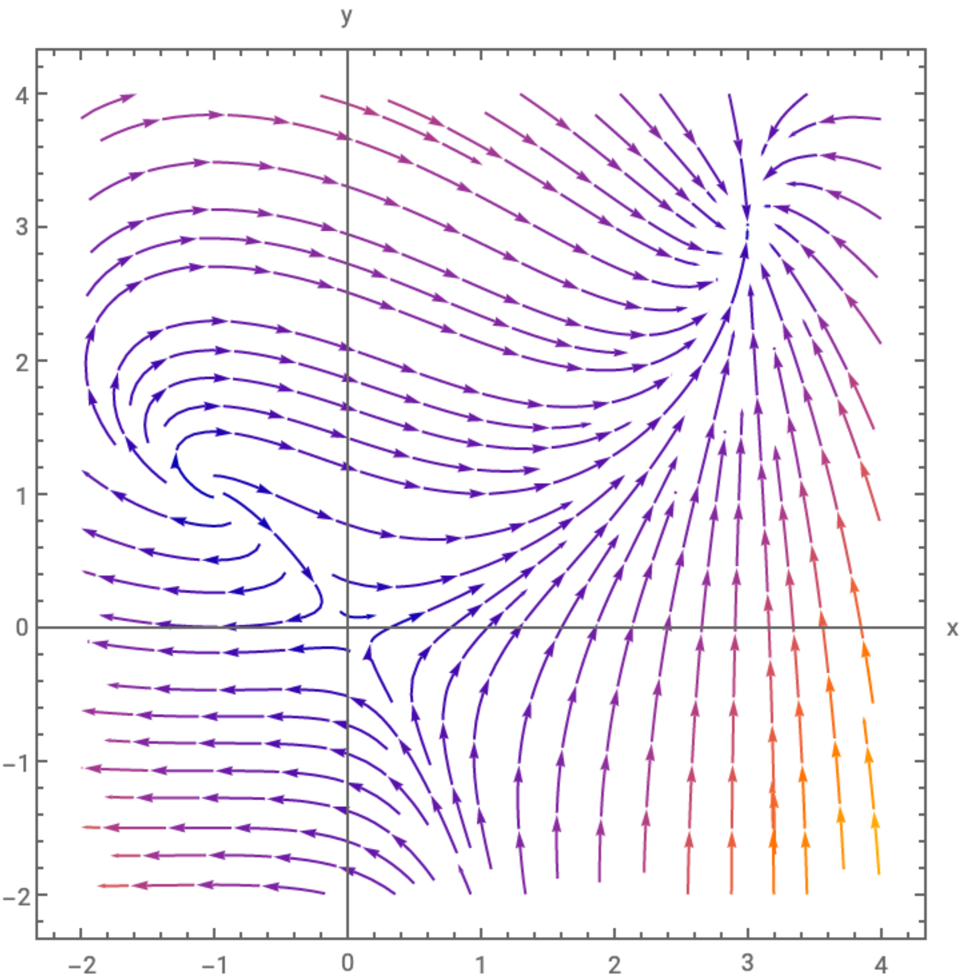
3) Near $(3,3)$: $J(3,3) = \begin{bmatrix} -6-3+3 & -3+3 \\ 6-3+1 & -3-1 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 4 & -4 \end{bmatrix}$

The associated linear system is $\vec{x}' = \begin{bmatrix} -6 & 0 \\ 4 & -4 \end{bmatrix} \vec{x}$.

We determine the eigenvalues of the associated linear system:

$\det(A - \lambda I) = \det\left(\begin{bmatrix} -6-\lambda & 0 \\ 4 & -4-\lambda \end{bmatrix}\right) = (-6-\lambda)(-4-\lambda) - 0 = 0$, $\lambda = -6, -4$. Both eigenvalues are real and negative. The behavior near $(3,3)$ is an asymptotically stable node (sink).

This problem will have a region of asymptotic stability. We can approximate it, if we take the time to calculate the eigenvectors, or through the phase portrait:



The region of asymptotic stability is nearly the entire plane. The two other equilibria will not approach $(3,3)$, but it appears all other points will approach the equilibrium at $(3,3)$. This is an example of something called **global asymptotic stability**, when all trajectories approach a single critical point.

■

6. Laplace Transforms

Transforms are operations that change a problem's coordinate system. Recall from Calculus III: Linear transformations. In that course, we converted problems in the domain of (x,y) to (u,v) to change the geometry to an area that was integrable. In doing this, we completed a linear transform defined using the change of coordinates and the determinant of a Jacobian matrix to "map" back to the original area in the integral.

Laplace transforms are a specific transformation mapping from t-space to s-space. We refer to s-space as **spectral space**, where s is a complex value of frequency which relates to the exponential form a complex value.

6.1 The Laplace Transform

The Laplace Transform is defined by

$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$

This transformation will always result in an equation of s alone.

The Laplace Transform is particularly useful in converting differential equations to algebraic forms that can be solved, which is why we introduce it in this course.

To apply it, there are some important properties of the Laplace Transform to understand and use.

Structure: The structure of the Laplace Transform can be used to expose the structure of a general transform (there are many.)

$F(s) = \int_\alpha^\beta f(t)K(t,s)dt$

In this form, the output is the transform of $f(t)$ in terms of the new variable and space, s. $K(t,s)$ is what we refer to as the **Kernel** of the transform - it is defining the mapping from t-space to s-space. (The Kernel of the Laplace Transform is e^{-st}.)

The primary use of these transformations is to simplify calculations by translating the problem to a new domain.

Linearity: The Laplace Transform is a linear operator, which means that it satisfies certain properties of a linear operator.

1. $L[f(x)+g(x)] = L[f(x)] + L[g(x)]$ When the linear operator is applied to the sum of two functions, the result is the sum of the linear operator on each function separately. This is also referred to as the *Principle of Superposition*.
2. $L[af(x)] = aL[f(x)]$ When the linear operator is applied to a function multiplied by a constant, the result is the same as that constant multiplied by the linear operator on the function alone.

These properties are often summarized into one expression $L[af(x)+bg(x)] = aL[f(x)] + bL[g(x)]$

We can show that a given operator is linear by testing these properties.

■ **Example 6.1** Show that $\frac{d}{dt}[\cdot]$ is a linear operator.

First, we can show the additive property:

$\frac{d}{dt}[f(t)+g(t)] = f'(t)+g'(t) = \frac{d}{dt}[f(t)] + \frac{d}{dt}[g(t)]$ When we complete a derivative of a sum, there isn't a rule that affects the sum, we simply apply the derivative operator to each function in the sum. Example: $\frac{d}{dt}[t+e^t] = \frac{d}{dt}[t] + \frac{d}{dt}[e^t] = 1+e^t$. (Note: the example is *not* sufficient to show linearity, it must be shown in general.)

Then, we can show the constant multiple property: $\frac{d}{dt}[af(t)] = \frac{d}{dt}[a]f(t) + a\frac{d}{dt}[f(t)] = a\frac{d}{dt}[f(t)]$. When we complete a derivative of a function multiplied by a constant, the derivative of the constant is zero, so the result is the same as the constant multiplied by the derivative of the function. ■

Use these properties, and the integral form of $\mathcal{L}[\cdot]$ to show yourself that the Laplace Transform is a Linear operator.

Discontinuous Functions: A Laplace Transform can be applied to a discontinuous function if it is **piecewise continuous**. For a function to be called piecewise continuous on an interval $[a,b]$, it must:

1. Have a finite number of discontinuities on $[a,b]$
2. For each discontinuity, $c \in [a,b]$, the one-sided limits must exist: $\lim_{t\to c-} f(t)$ and $\lim_{x\to c+} f(t)$ (no infinite discontinuities.)

In short, *jump discontinuities* and *point discontinuities* (holes) are acceptable, but *infinite discontinuities* (asymptotes) are not acceptable.

When a jump discontinuity or point discontinuity exist in the function $f(t)$, we treat it as in Calculus I - by splitting up the interval.

■ **Example 6.2** Given a piecewise continuous function $f(t) = \begin{cases} 1, & t<5, \\ 2, & 5 \le t. \end{cases}$

Determine the Laplace Transform of $f(t)$.

$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$

Due to the jump discontinuity at $t=5$, we must split up the integral into two pieces:

$\int_0^\infty f(t)e^{-st}dt = \int_0^5 1e^{-st}dt + \int_5^\infty 2e^{-st}dt$

$\int_0^5 e^{-st}dt = \frac{-1}{s}e^{-st}|_0^5 = \frac{-1}{s}\left(e^{-5s}-1\right)$

$\int_5^\infty 2e^{-st}dt = \lim_{b\to\infty}\int_5^b 2e^{-st}dt = \lim_{b\to\infty}\frac{-2}{s}e^{-st}|_5^b = \lim_{b\to\infty}\frac{-2}{s}\left(e^{-bs}-e^{-5s}\right) = \frac{-2}{s}\left(0-e^{-5s}\right).$

Adding the two together, we have $\mathcal{L}[f(t)] = \frac{-1}{s}\left(e^{-5s}-1\right) + \frac{-2}{s}\left(0-e^{-5s}\right) = \frac{1}{s}(e^{-5s}+1)$. ■

A piecewise continuous function which commonly appears in applications is the Heaviside step function, $u(t)$.

$u(t) = \begin{cases} 0, & t<0, \\ 1, & 0 \le t. \end{cases}$

For practice, let's apply the Laplace Tranform to a few functions.

■ **Example 6.3** Determine the Laplace Transform of $f(t) = 13t$.
$\mathcal{L}[f(t)] = \int_0^\infty 13te^{-st}dt$
This will require the use of integration by parts.

u	dv
$13t$	$e^{-st}dt$
13	$\frac{-1}{s}e^{-st}$
0	$\frac{1}{s^2}e^{-st}$

$\int_0^\infty 13te^{-st}dt = \lim_{b\to\infty}\int_0^b 13te^{-st}dt = \lim_{b\to\infty}\left(\frac{-13t}{s}e^{-st} - \frac{13}{s^2}e^{-st}\right)|_0^b = \lim_{b\to\infty}\left(\frac{-13t}{s}e^{-sb} - \frac{13}{s^2}e^{-sb} - \left(\frac{-13}{s}\right.\right.$
$\left(0-0+0+\frac{13}{s^2}\right) = \frac{13}{s^2}$
$F(s) = \frac{13}{s^2}$ ■

Note: This is the same as $13\times\mathcal{L}[t]$, see table of Laplace Transforms for common forms.

■ **Example 6.4** Determine the Laplace Transform of $f(t) = 2t + 5e^{3t}$.
$\mathcal{L}[f(t)] = \int_0^\infty \left(2t+5e^{3t}\right)e^{-st}dt$
We can break this into two terms to compute the Laplace Transform.
$\int_0^\infty \left(2t+5e^{3t}\right)e^{-st}dt = \int_0^\infty 2te^{-st}dt + \int_0^\infty 5e^{3t}e^{-st}dt$
The first term will require integration by parts, the second will require u-substitution.
For $\int_0^\infty 2te^{-st}dt$

u	dv
$2t$	$e^{-st}dt$
2	$\frac{-1}{s}e^{-st}$
0	$\frac{1}{s^2}e^{-st}$

$\lim_{b\to\infty}\int_0^b 2te^{-st}dt = \lim_{b\to\infty}\left(\frac{-2t}{s}e^{-st} - \frac{2}{s^2}e^{-st}\right)|_0^b = \lim_{b\to\infty}\left(\frac{-2b}{s}e^{-sb} - \frac{2}{s^2}e^{-sb} + \frac{-2*0}{s}e^0 + \frac{2}{s^2}e^0\right) =$
$\frac{2}{s^2}$.
For $\int_0^\infty 5e^{3t}e^{-st}dt$ we need to use algebra to rewrite it as one exponential.
$\int_0^\infty 5e^{3t}e^{-st}dt = \int_0^\infty 5e^{(3-s)t}dt = \lim_{b\to\infty}\int_0^b 5e^{(3-s)t}dt = \lim_{b\to\infty}\frac{5}{3-s}e^{(3-s)t}|_0^b = \lim_{b\to\infty}\frac{5}{3-s}\left(e^{(3-s)b} - e^{(3-s)*0}\right) =$
$\frac{5}{3-s}(0-1) = \frac{5}{s-3}$
Then, $\mathcal{L}[f(t)] = \frac{2}{s^2} + \frac{5}{s-3}$. ■

■ **Example 6.5** Determine the Laplace Transform of $f(t) = 3\cos(2t)$.
$\mathcal{L}[f(t)] = \int_0^\infty 3\cos(2t)e^{-st}dt$.
While we are capable of using integration by parts to solve this expression - it isn't necessary.
It's much easier to rewrite $\cos(2t)$ in terms of complex exponentials.
Recall: $e^{ait} = \cos(at) + i\sin(at)$ and $e^{-ait} = \cos(at) - i\sin(at)$.
Thus, $\cos(2t) = \frac{e^{2it}+e^{-2it}}{2}$
Then $\int_0^\infty 3\cos(2t)e^{-st}dt = \int_0^\infty\left(\frac{e^{2it}+e^{-2it}}{2}\right)e^{-st}dt = \int_0^\infty\frac{1}{2}\left(e^{(2i-s)t}+e^{(-2i-s)t}\right)dt = \lim_{b\to\infty}\int_0^b\frac{1}{2}\left(e^{(2i-s)t}+e^{(-2i-\right.$
$\lim_{b\to\infty}\frac{1}{2}\left(\frac{1}{2i-s}e^{(2i-s)t} + \frac{1}{-2i-s}e^{(-2i-s)t}\right)|_0^b = \lim_{b\to\infty}\frac{1}{2}\left(\frac{1}{2i-s}e^{(2i-s)b} + \frac{1}{-2i-s}e^{(-2i-s)b} - \frac{1}{2i-s}e^{(2i-s)*0} - \frac{}{-2}\right.$

$$\frac{1}{2}\left(0+0-\frac{1}{2i-s}-\frac{1}{-2i-s}\right)=\frac{1}{2}\left(\frac{-2i+s+2i+s}{(2i-s)(-2i-s)}\right)=\frac{s}{4+s^2}.$$

So, $\{() = \frac{s}{4+s^2}$. ■

6.1.1 Existence and Uniqueness of Laplace Transforms

Notice that in the limits for the infinite bounds, we're effectively asking ourselves "Which function grows faster as $b \to \infty$?"

If $f(t)$ grows "slower" than e^{-st} decays as $t \to \infty$, then the integral converges and we can define the transform.

If $f(t)$ grows "faster" than e^{-st} decays as $t \to \infty$, then the integral diverges. We cannot define a Laplace Transform in this case. An example of this situation would be a function like $f(t) = te^{t^2}$.

If we evaluate the limit $\lim_{t\to\infty} \frac{f(t)}{e^{st}}$ and the limit exists and is finite - the transform will exist. If the limit diverges, the transform will not exist.

In the case we can pick a specific value that causes the limit to converge, we say $f(t)$ is of exponential order.

Generally speaking, we seek for $f(t)$ to be exponentially bounded to ensure the Laplace Transform exists.

$f(t)$ is exponentially bounded if $|f(t)| \le Me^{at}$ for $M \ge 0$ and a constant a. This means that the graph of $f(t)$ will always lie between the graphs of $-Me^{at}$ and Me^{at}.

Consistency: If two piecewise continuous, exponentially bounded functions have the same Laplace Transform, the functions must also be equal for $t \ge 0$. We'll get into this more when we determine inverse Laplace Transforms.

6.1.2 Inverse Laplace Transforms

In order to solve problems, we need to find both the Laplace Transforms of functions, and their inverse Laplace Transforms.

The inverse Laplace Transfrom allows us to transform back to t from s. Two functions with the same transform must be the same at all values where they are piecewise continuous.

Theorem 6.1.1 If $f(t)$ is piecewise continuous and of exponential order on $[0,\infty)$ and $\mathcal{L}[f(t)] = F(s)$, then $f(t) = \mathcal{L}^{-1}[F(s)]$ where $\mathcal{L}^{-1}[\cdot]$ is the inverse Laplace Transform.

Note: $\mathcal{L}^{-1}[\cdot]$ is also a linear operator. $\mathcal{L}^{-1}[c_1F_1(s)+C_2F_2(s)] = c_1\mathcal{L}^{-1}[F_1(s)]+c_2\mathcal{L}^{-1}[F_2(s)]$.

In order to use an inverse Laplace Transform, reference a table of Laplace Transforms - match the appropriate form and determine the function consistent in form. We do not mention the integral form, as that is beyond this course.

Table of Common Laplace Transforms

$f(t) = \mathcal{L}^{-1}(F(s))$	$F(s) = \mathcal{L}(f(t))$
1	$1/s$, $s > 0$
e^{at}	$1/(s-a)$, $s > a$
t^n, $n \in \mathbb{N}$	$n!/s^{n+1}$, $s > 0$
t^a, $a > -1$	$\Gamma(a+1)/s^{a+1}$, $s > 0$
$\sin at$	$a/(s^2+a^2)$, $s > 0$
$\cos at$	$s/(s^2+a^2)$, $s > 0$
$\sinh at$	$a/(s^2-a^2)$, $s > \lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$, $s > \lvert a\rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$, $s > a$
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$, $s > a$
$t^n e^{at}$, $n \in \mathbb{N}$	$n!/(s-a)^{n+1}$, $s > a$
$u_c(t)$	e^{-cs}/s, $s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$(1/c)F(s/c)$, $c > 0$
$\delta(t-c) = \delta_c(t)$	e^{-cs}
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

■ **Example 6.6** Determine the inverse Laplace Transform for $F(s) = \dfrac{2}{s^2+4} + \dfrac{36}{s^4}$

Since the inverse Laplace Transform is a linear operator, we can evaluate each term separately, $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\dfrac{2}{s^2+4}\right] + \mathcal{L}^{-1}\left[\dfrac{36}{s^4}\right]$

The first term $\dfrac{2}{s^2+4}$ looks like the term in the table $\dfrac{a}{s^2+a^2}$, with $a = 2$, so $\mathcal{L}^{-1}\left[\dfrac{2}{s^2+4}\right] = \sin(2t)$.

The second term $\dfrac{36}{s^4}$ looks most like $\dfrac{n!}{s^{n+1}}$ with $n = 3$. However, $3! = 6$, not 36. So, we have an additional multiple of 6. Then $\mathcal{L}^{-1}\left[\dfrac{36}{s^4}\right] = 6t^3$.

Combined, $\mathcal{L}^{-1}[F(s)] = \sin(2t) + 6t^3$.

■

Terms that look exactly like the table are easier to match up, but there are also cases where we have products of terms that we need to separate using partial fraction decomposition.

■ **Example 6.7** Determine the inverse Laplace Transform for $F(s) = \dfrac{2+s^2}{(1+s^2)(s^2+4s+4)}$.

Here we have a single term, but it doesn't match any of the ones in the table - it's far more complex. So, we need to expand the terms in order to use the table.

First, we need to recognize that $1+s^2 = s^2+1$ which looks like s^2+a^2 in the table.

Second, we need to recognize that $s^2+4s+4 = (s+2)^2$ which looks like $(s-a)^2$ in the table.

Third, we need to use partial fraction decomposition to decompose this fraction into multiple separate fractions with those denominators.

The first term is referred to as an *irreducible quadratic*, and we will seek a fraction of the form $\dfrac{As+B}{s^2+a^2}$.

The second term is a repeated linear factor (it's squared). We will seek two terms in its factorization: $\dfrac{C}{s-a} + \dfrac{D}{(s-a)^2}$.

Combining these, we will seek the values of A, B, C, and D such that $\dfrac{2+s^2}{(s^2+1)(s+2)^2} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s+2} + \dfrac{D}{(s+2)^2}$

We solve by multiplying both sides by the denominator on the left:

$2+s^2 = (As+B)(s+2)^2 + C(s^2+1)(s+2) + D(s^2+1) = As^3+Bs^2+4As^2+4Bs+4As+4B+Cs^3+2Cs^2+Cs+2C+Ds^2+D$

Then, we can match them by their respective powers of s:

s^3 terms: $0 = A+C$

s^2 terms: $1 = B+4A+2C+D$

s terms: $0 = 4B+4A+C$

Constant terms: $2 = 4B+2C+D$

We have four equations and four unknowns: solve algebraically for $A = \dfrac{-4}{25}$, $B = \dfrac{3}{25}$, $C = \dfrac{4}{25}$, and $D = \dfrac{30}{25}$.

Substitute the coefficients into the form of solution: $\dfrac{2+s^2}{(s^2+1)(s+2)^2} = \dfrac{\frac{-4}{25}s+\frac{3}{25}}{s^2+1} + \dfrac{\frac{4}{25}}{s+2} + \dfrac{\frac{30}{25}}{(s+2)^2} = \dfrac{1}{25}\left(\dfrac{-4s+3}{s^2+1} + \dfrac{4}{s+2} + \dfrac{30}{(s+2)^2}\right)$

We need to do one more expansion step - separate the terms in the numerator of the first term of our expanded form.

$$\frac{2+s^2}{(s^2+1)(s+2)^2} = \frac{1}{25}\left(\frac{-4s}{s^2+1} + \frac{3}{s^2+1} + \frac{4}{s+2} + \frac{30}{(s+2)^2}\right)$$

Now each term looks like a piece in the table.

The first term $\dfrac{-4s}{s^2+1}$ looks like $\dfrac{s}{s^2+a^2}$ with $a = 1$ and a constant multiple of -4. So, $\mathcal{L}^{-1}\left[\dfrac{-4s}{s^2+1}\right] = -4\cos(t)$.

The second term $\dfrac{3}{s^2+1}$ looks like $\dfrac{a}{s^2+a^2}$ with $a = 1$ and a constant multiple of 3. So, $\mathcal{L}^{-1}\left[\dfrac{3}{s^2+1}\right] = 3\sin(t)$.

The third term $\frac{4}{s+2}$ looks like $\frac{1}{s-a}$ with $a=-2$ and a constant multiple of 4. So, $\mathcal{L}^{-1}\left[\frac{4}{s+2}\right]=4e^{-2t}$.

The fourth term $\frac{30}{(s+2)^2}$ looks like $\frac{n!}{(s-a)^{n+1}}$ with $a=-2$, $n=1$, and a constant multiple of 30. So,$\mathcal{L}^{-1}\left[\frac{30}{(s+2)^2}\right]=30te^{-2t}$.

Combining the results, $\mathcal{L}^{-1}[F(s)]=\frac{1}{25}\left(-4\cos(t)+3\sin(t)+4e^{-2t}+30te^{-2t}\right)$. (Looks a lot like our solutions to earlier problems! It is actually the solution to $x''+4x'+4x=\sin(t)$ with $x(0)=0$ and $x'(0)=1$.) ■

With the skills of calculating Laplace Transforms and their inverses, we can use them to solve ODEs.

6.2 Solving Initial Value Problems

The most powerful property of the Laplace Transform is the result of its operation on the derivative of a function. The Laplace Transform converts derivative operations to algebraic relations.

$\mathcal{L}[f'(t)]=s\mathcal{L}[f(t)]-f(0)$

This operation extends to higher order derivatives,

$\mathcal{L}[f''(t)]=s\mathcal{L}[f'(t)]-f'(0)=s\left(s\mathcal{L}[f(t)]-f(0)\right)-f'(0)=s^2\mathcal{L}[f(t)]-sf(0)-f'(0)$

This converts the problem to an algebra one, which we can then use to solve for $F(s)$. Once we have $F(s)$, we invert the transform and recover the solution to the ODE.

■ **Example 6.8** Solve $x''+4x'+4x=\sin(t)$ with $x(0)=0$ and $x'(0)=1$ using Laplace transforms.

We apply a Laplace Transform to the entire equation. However, since it is a linear operator, it may be easier to evaluate each term separately.

$\mathcal{L}[x'']=s^2\mathcal{L}[x]-sx(0)-x'(0)=s^2\mathcal{L}[x]-1$

$\mathcal{L}[4x']=4(s\mathcal{L}[x]-x(0))=4s\mathcal{L}[x]$

$\mathcal{L}[4x]=4\mathcal{L}[x]$

$\mathcal{L}[\sin(t)]=\frac{1}{s^2+1)}$

So, the transformed equation is then $s^2\mathcal{L}[x]-1+4s\mathcal{L}[x]+4\mathcal{L}[x]=\frac{1}{s^2+1)}$

We then solve for $\mathcal{L}[x]$, which is often shorthanded by using a capital letter for the function $\mathcal{L}[x]=X$.

$(s^2+4s+4)X=\frac{1}{s^2+1)}+1=\frac{s^2+2}{s^2+1}$

So $X=\frac{s^2+2}{(s^2+1)(s^2+4s+4)}$, which is the exact function in the previous example.

We determine the inverse Laplace Transform of this result to determine the solution to the initial value problem.

As we found in the last example, this is $\frac{1}{25}\left(-4\cos(t)+3\sin(t)+4e^{-2t}+30te^{-2t}\right)$.

■

To summarize the steps:

1. Apply a Laplace Transform to the entire equation (use the property for derivatives!)
2. Solve for the function in s-space, $(X(s))$.
3. Apply an inverse Laplace Transform to determine the solution, $x(t)$. (Use partial fraction decomposition as needed.)

■ **Example 6.9** Determine the solution of $x'' + 2x' + x = \cos(2t)$, $x(0) = 1$, and $x'(0) = 2$, using Laplace Transforms.

First, we apply a Laplace Transform to the entire equation:

$\mathcal{L}[x''] = s^2X - sx(0) - x'(0) = s^2X - s - 2$

$\mathcal{L}[2x'] = 2(sX - x(0)) = 2sX - 2$

$\mathcal{L}[x] = X$

$\mathcal{L}[\cos(2t)] = \dfrac{s}{s^2+4}$

Then, $s^2X - s - 2 + 2sX - 2 + X = \dfrac{s}{s^2+4}$

$(s^2+2s+1)X = \dfrac{s}{s^2+4} + s + 4 = \dfrac{s + s^3 + 4s + 4s^2 + 16}{s^2+4}$

$X = \dfrac{s^3 + 4s^2 + 5s + 16}{(s^2+4)(s+1)^2}$

Then, we need to separate the terms to apply an inverse Laplace Transform. We seek to rewrite $\dfrac{s^3+4s^2+5s+16}{(s^2+4)(s+1)^2} = \dfrac{As+B}{s^2+4} + \dfrac{C}{s+1} + \dfrac{D}{(s+1)^2}$

Multiply both sides by the denominator on the left, then match terms.

$s^3 + 4s^2 + 5s + 16 = (As+B)(s+1)^2 + C(s^2+4)(s+1) + D(s^2+4) = As^3 + Bs^2 + 2As^2 + 2Bs + As + B + Cs^3 + Cs^2 + 4Cs + 4C + Ds^2 + 4D$

s^3 terms: $1 = A + C$

s^2 terms: $4 = B + 2A + C + D$

s terms: $5 = 2B + A + 4C$

Constant terms: $16 = B + 4C + 4D$

Solve algebraically for $A = \dfrac{-3}{25}$, $B = \dfrac{8}{25}$, $C = \dfrac{28}{25}$, and $D = \dfrac{70}{25}$.

Then, $X(s) = \dfrac{1}{25}\left(\dfrac{-3s+8}{s^2+4} + \dfrac{28}{s+1} + \dfrac{70}{(s+1)^2}\right)$. Separate the terms in the numerator of the first term, and then apply an inverse Laplace Transform.

$X(s) = \dfrac{1}{25}\left(\dfrac{-3s}{s^2+4} + \dfrac{8}{s^2+4} + \dfrac{28}{s+1} + \dfrac{70}{(s+1)^2}\right)$.

$\mathcal{L}^{-1}\left[\dfrac{-3s}{s^2+4}\right] = -3\cos(2t)$

$\mathcal{L}^{-1}\left[\dfrac{8}{s^2+4}\right] = 8\sin(2t)$

$\mathcal{L}^{-1}\left[\dfrac{28}{s+1}\right] = 28e^{-t}$

$\mathcal{L}^{-1}\left[\dfrac{70}{(s+1)^2}\right] = 70te^{-t}$

So $x(t) = \dfrac{1}{25}\left(-3\cos(2t) + 8\sin(2t) + 28e^{-t} + 70te^{-t}\right)$

■

6.3 Delta Functions and Forcing

We discussed the Laplace Transform of piecewise functions earlier, in this section we will expand a little and focus on the Heaviside step function and delta function transforms when solving ODEs.

Recall: the Heaviside step function is defined $u(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t \end{cases}$

Additionally, $u(t-c) = u_c(t) = \begin{cases} 0, & t < c \\ 1, & c \leq t \end{cases}$

We can combine terms of the Heaviside step functions in order to create square pulse functions. (These are particularly good for modeling switches - 0 for off, 1 for on.) We can also use them to define any piecewise function through Heaviside step functions.

A square wave may be like: $u_0 - u_4 = \begin{cases} 0, & t < 0 \\ 1, & 0 \leq t < 4 \\ 0, & 4 \leq t \end{cases}$

We can shorthand this as $u_{04}(t)$

■ **Example 6.10** Rewrite the piecewise function $f(t) = \begin{cases} 0, & t < 0t^2, \quad 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ e^t, & 2 \leq t \end{cases}$ in terms of Heaviside step functions.

The function is zero to the left of $t = 0$, so not much to do there.

The function is t^2 between 0 and 1, so we construct the pulse function u_{01} and multiply it by t^2.

The function is 1 between 1 and 2, so we construct the pulse function u_{12} and multiply it by 1.

The function is e^t from 2 on, so we use u_2 and multiply it by e^t.

Then, $f(t) = t^2 u_{01} + u_{12} + e^t u_2$.

This is the same as $f(t) = t^2 (u_0 - u_1) + (u_1 - u_2) + e^t u_2 = t^2 u_0 + (1 - t^2) u_1 + (e^t - 1) u_2$. ■

The Laplace Transform of a Heaviside step function is given by $\mathcal{L}[u_c(t)] = \dfrac{e^{-cs}}{s}$.

The Laplace Transform of a product of a function with the Heaviside step function is given by $\mathcal{L}[u_c(t) f(t)] = e^{-cs} \mathcal{L}[f(t)] = e^{-cs} F(s)$. Similarly, when applying an inverse Laplace Transform, $\mathcal{L}[e^{-cs} F(s)] = u_c(t) f(t - c)$.

■ **Example 6.11** Determine the Laplace Transform of the piecewise function from the last example.

$\mathcal{L}[t^2 u_0 + (1 - t^2) u_1 + (e^t - 1) u_2] = e^{0s} \mathcal{L}[t^2] + e^{-s} \mathcal{L}[1 - t^2] + e^{-2s} \mathcal{L}[e^t - 1]$.

$\mathcal{L}[t^2] = \dfrac{2!}{s^3} = \dfrac{2}{s^3}$

$\mathcal{L}[1 - t^2] = \dfrac{1}{s} - \dfrac{2}{s^3}$

$\mathcal{L}[e^t - 1] = \dfrac{1}{s - 1} - \dfrac{1}{s}$

So, the Laplace Transform simplifies to $\dfrac{2}{s^3} + e^{-s} \left(\dfrac{1}{s} - \dfrac{2}{s^3} \right) + e^{-2s} \left(\dfrac{1}{s - 1} - \dfrac{1}{s} \right)$ ■

6.3.1 The Dirac Delta Function

The Dirac delta function is a function with a pulse of infinitesimal width. It represents a concept of a compressed pulse which has an area under the curve of 1 in a limiting case approaching a point-width pulse. Some describe it as a function with no width and infinite height, but the area under the curve is always 1.

The Dirac delta function is an impulse function, it represents a very rapid force on the system (when used as a forcing function.)

We often represent the Dirac delta function through $\delta(t)$, where $\delta(t) = 0$ for all values of $t \neq 0$. We do not have a defined value at $t = 0$, but instead refer to the property of its area $\int_{-\infty}^{\infty} \delta(t) dt = 1$. This is saying that it provides an impulse of 1 at $t = 0$.

We can also shift this function, as we did for the Heaviside step function.

$\delta(t - c) = \delta_c(t)$, and in these cases $\delta_c(t) = 0$ for $t \neq c$

The Laplace transform of the Dirac delta function is given by $\mathcal{L}[\delta_c(t)] = e^{-cs}$, which - you may see that the Laplace transform is 1 when $c = 0$.

■ **Example 6.12** Solve the differential equation $x'' + 6x' + 5x = \delta(t)$ where $x(0) = 0$ and $x'(0) = 1$ using Laplace Transforms.

First, we apply a Laplace Transform to the entire equation.

$s(sX - 0) - 1 + 6(sX - 0) + 5X = 1$

$s^2X - 1 + 6sX + 5X = 1$

$(s^2 + 6s + 5)X = 2$

$X(s) = \dfrac{2}{s^2 + 6s + 5} = \dfrac{2}{(s+5)(s+1)}$

Use partial fraction decomposition to write this as two linear terms.

$\dfrac{2}{(s+5)(s+1)} = \dfrac{A}{s+5} + \dfrac{B}{s+1}$

$2 = A(s+1) + B(s+5) = As + Bs + A + 5B$

s terms: $0 = A + B$, $A = -B$

constant terms: $2 = A + 5B$, $2 = 4B$, so $B = \dfrac{1}{2}$ and $A = \dfrac{-1}{2}$

$X(s) = \dfrac{1}{2}\left(\dfrac{-1}{s+5} + \dfrac{1}{s+1}\right)$

Determine the inverse Laplace transforms of each term.

$x(t) = \dfrac{1}{2}\left(-e^{-5t} + e^{-t}\right)$ ■

■ **Example 6.13** Solve the differential equation $x'' + 2x' + x = \delta_2(t)$ where $x(0) = 1$ and $x'(0) = 0$ using Laplace Transforms.

First, we apply a Laplace Transform to the entire equation.

$s(sX - 1) - 0 + 2(sX - 1) + X = e^{-2s}$

$s^2X - s + 2sX - 2 + X = e^{-2s}$

$(s^2 + 2s + 1)X = s + 2 + e^{-2s}$

$X = \dfrac{s}{(s+1)^2} + \dfrac{2}{(s+1)^2} + \dfrac{e^{-2s}}{(s+1)^2}$

Apply an inverse Laplace Transform to each term. However, the first term doesn't work out, so we need to manipulate the first two terms.

$X = \dfrac{s+1}{(s+1)^2} + \dfrac{1}{(s+1)^2} + \dfrac{e^{-2s}}{(s+1)^2} = \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{e^{-2s}}{(s+1)^2}.$

Then, the first term matches the form of e^{at} with $a = -1$.

The second term appears of the form of a $t^n e^{at}$ with $a = -1$ and $n = 1$.

The third term has an e^{-cs} multiplied by the second term, this results in a shift with a $u_c(t)$.

$x(t) = e^{-t} + te^{-t} + u_2(t)(t-2)e^{-t+2}$ ■

The solution using a delta function is also referred to as a Green's function solution. We will explore this topic further in PDEs.