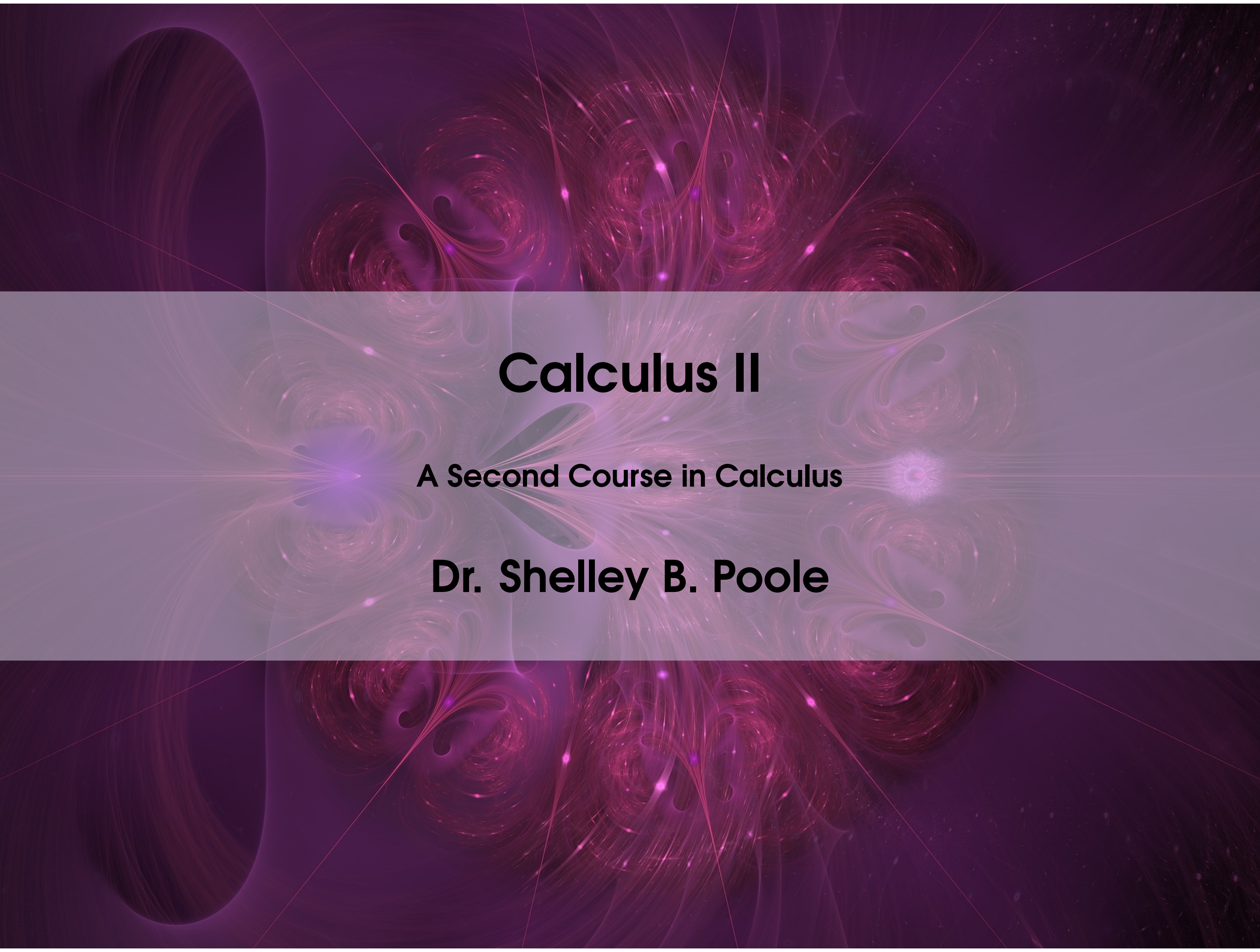

Calculus II

A Second Course in Calculus

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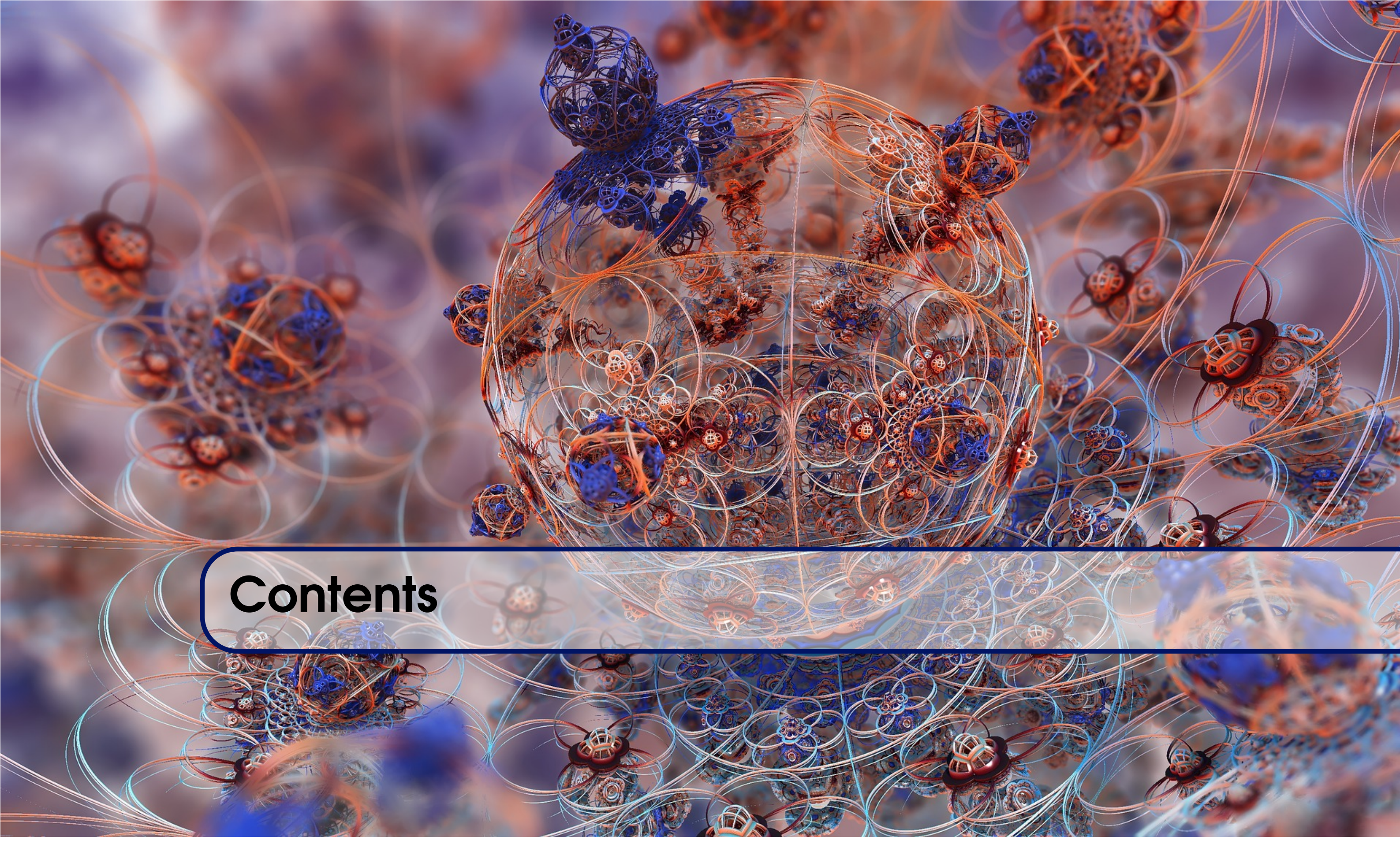

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1. Introduction

1.1 Disclaimer

These are an experiment. They won't be perfect, and they are not yet complete. The formatting will be weird sometimes, and there may be gaps, typos, or other errors in the text. I'll also add notes to myself in [red] for further editing. I welcome your feedback to improve these notes, but I may not be able to implement everything as we progress through the course. Thanks for reading these, and I'll see you in class!

1.2 Introduction

Calculus II builds upon the material you learned in Calculus I. You will need to recall several concepts from Algebra, Trigonometry, and Calculus I. I am an applied mathematician, so I teach this material in the context of its physical representations in space, and the applications you might encounter as a scientist. I want to see that you understand what you are calculating, not just that you *can* calculate things. These notes are meant to fill in gaps so you can solidify concepts and grasp the material. Please let me know if you have questions!

1.3 Algebra, Trigonometry, and Calculus I Review list

These are topics that you should be comfortable with upon entering this course. If there is anything on this list that you need help with, please SEE ME SOON. Neither of us wants you to fall behind.

1. Factoring (Algebra)
2. Completing the Square (Algebra)
3. Quadratic Formula (Algebra)
4. Properties of Logarithms (Algebra)
5. Exponential growth and population models (Algebra)
6. Logistic Models (Algebra)
7. Area of a sector of a circle (Trigonometry)
8. Values of Trigonometric functions based on a point (Trigonometry)
9. Trigonometric Identities, and solving for function values using identities (Trigonometry)

10. Unit circle and trigonometric function values in Radians (Trigonometry)
11. Determining an angle from a trigonometric function value (Trigonometry)
12. Values of inverse trigonometric functions (Trigonometry)
13. Polar Coordinates (Trigonometry)
14. Polar equations (Trigonometry)
15. Parametric curves (Trigonometry)
16. Derivatives, all rules and implicit differentiation (Calc I)
17. Tangent lines (Calc I)
18. Projectile motion problems (Calc I)
19. Antiderivatives (Calc I)
20. Summation notation, Riemann Sums (Calc I)
21. Fundamental Theorem of Calculus (Calc I)
22. Properties of Integrals, even and odd functions, etc (Calc I)
23. U-Substitution (Calc I)
24. Limits, regular and at infinity (Calc I)
25. L'Hopital's rule (Calc I)

2. Review

2.1 Fundamental Theorem of Calculus

Theorem 2.1.1 If f is continuous on $[a,b]$, and F is any antiderivative of f on $[a,b]$, then we can define $\int_a^b f(x)dx = F(b) - F(a)$

■ **Example 2.1** Determine the area under the curve $f(x) = 2 + x - 2x^2$ between $x = 1$ and $x = 2$

$\int_1^2 \left(2 + x - 2x^2\right)$

The antiderivative of $f(x)$ is $F(x) = 2x + \frac{x^2}{2} - \frac{2x^3}{3} + C$

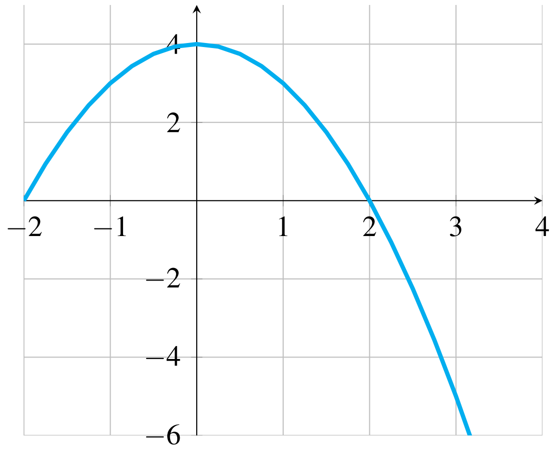
So, $\int_1^2 \left(2 + x - 2x^2\right) = F(2) - F(1) = \left(2*2 + \frac{2^2}{2} - \frac{2*2^3}{3}\right) - \left(2*1 + \frac{1^2}{2} - \frac{2*1^3}{3}\right) = \left(4 + 2 - \frac{16}{3}\right) - \left(2 + \frac{1}{2} - \frac{2}{3}\right) = \frac{2}{3} - \frac{11}{6} = \frac{-7}{6} units^2$ ■

2.1.1 Area Functions

Recall: $A(x) = \int_a^x f(t)dt$ yields an area function for the area under function $f(t)$ on the interval $a \leq t \leq x$.

■ **Example 2.2** Given $f(t) = 4 - t^2$, define the area function for the area under $f(t)$ between $a = 1$ and x

$$A(x) = \int_1^x \left(4 - t^2\right) dt = 4t - \frac{t^3}{3}\Big|_1^x = \left(4x - \frac{x^3}{3}\right) - \left(4 - \frac{1}{3}\right) = 4x - \frac{x^3}{3} - \frac{11}{3}$$



■

This function allows us to define the area for multiple intervals easily.

What is the area under the curve from $x = 1$ to $x = 2$? Since we used $a = 1$ in the definition of $A(x)$, we just need to evaluate $A(2) = 8 - \frac{8}{3} - \frac{11}{3} = \frac{5}{3} units^2$.

What is the area under the curve from $x = 1$ to $x = 3$? We evaluate $A(3) = 12 - 9 - \frac{11}{3} = \frac{-2}{3} units^2$.

This saves us the hassle of integrating the same function multiple times to evaluate the area under the curve more than once.

Theorem 2.1.2 If f is continuous on $[a,b]$, then $A(x)$ is continuous on $[a,b]$ and differentiable on (a,b) such that $A'(x) = f(x)$.

We can verify this with our example problem: $\frac{d}{dx}\left[4x - \frac{x^3}{3} - \frac{11}{3}\right] = 4 - x^2 - 0 = 4 - x^2$ ✓

2.2 Substitution Rule

Generally shortened to “U-sub", this should be your go-to integration technique for functions that do not have a known antiderivative. It will not work for all integrals, but it is the simplest integration rule you will learn.

U-substitution “undoes" the chain rule. Recall: Chain rule $\frac{d}{dx}[f(g(x))] = \frac{df}{dx}(g(x)) \times \frac{dg}{dx}(x)$ or $f'(g(x)) \times g'(x)$

In order to apply your U-substitution rule, you must first identify the internal function in the composition, $g(x)$. You can recognize $g(x)$ if it is already inside of another function in the integral (although sometimes it is not). However, you will always need to see the derivative, $g'(x)$ in some form inside of integral.

Additionally, you will need to rewrite the entire integral in terms of the new variable u. No x's can remain!

To apply the u-substitution rule, you:

1. Set $u = g(x)$
2. Take the derivative of u, $\frac{du}{dx} = g'(x)$, which leads to a relation between du and dx
3. Replace dx with $\frac{du}{g'(x)}$ ensuring that you only have terms of u remaining
4. Replace all x terms with the equivalent forms in terms of u - this requires algebraic manipulation!
5. Integrate the resulting integral in terms of u just as you would with x - use the appropriate rules!
6. Convert your answer back into equivalent forms using $u = g(x)$ so you have an answer in terms of x.

General form: $\int f(g(x))g'(x)dx \to \int f(u)du = F(u) + C = F(g(x)) + C$ so long as $f(x)$ and $g(x)$ are continuous functions.

■ **Example 2.3** Integrate $\int e^{7x}dx$ using u-substitution

Note that $7x$ is <u>inside</u> the function e^x. The derivative of $7x$, $\frac{d}{dx}[7x] = 7$, is a constant. So, we can apply this u-substitution because $dx = \frac{du}{7}$. This yields a new integral:

$\int e^u \frac{du}{7} = \frac{1}{7}\int e^u du$

This looks just like $\frac{1}{7}\int e^x dx$ (Recall: the variable here is a placeholder, so whether it is x, y, u, or *bob* - it behaves the same way when we apply an antiderivative).

The antiderivative is $\frac{1}{7}e^u + C$

Converting it back to terms of x yields $\frac{1}{7}e^{7x} + C$ ■

This process adds another layer when we have a definite integral: now we have to update the bounds to be in terms of the same variable as our integral.

■ **Example 2.4** Integrate $\int_1^2 \frac{\ln x}{x}dx$ using u-substitution

Note that the derivative of $\ln x$, $\frac{d}{dx}[\ln x] = \frac{1}{x}$, so our $g(x) = \ln x$, and we want to define $u = \ln x$.

Define $\frac{du}{dx} = \frac{1}{x}$, so $dx = xdu$.

Substituting both of these into the integral yields: $\int \frac{u}{x}xdu = \int udu$

However, we haven't dealt with the bounds yet. We update the bounds by using our definition of $u = \ln x$. The lower bound is $x = 1$, so we determine what the value of u is when $x = 1$. $u = \ln(1) = 0$. We repeat this process for the upper bound $x = 2$, $u = \ln(2)$.

This yields $\int_0^{\ln(2)} udu$. Applying the Fundamental Theorem of Calculus yields $\frac{u^2}{2}\Big|_0^{\ln(2)} = \frac{(\ln(2))^2}{2} - \frac{0^2}{2} = \frac{1}{2}(\ln(2))^2$

We can verify this result by finding the antiderivative using u-substitution, and then applying the fundamental theorem of calculus.
$\int u du = \frac{u^2}{2} + C$ Converted back into a function of x: $F(x) = \frac{(\ln x)^2}{2} + C$

$F(2) - F(1) = \frac{(\ln(2))^2}{2} - \frac{(\ln(1))^2}{2} = \frac{1}{2}(\ln(2))^2$ (Same answer, slightly different approach) ■

3. Applications of Integration

3.1 Velocity and Net Change

This section discusses the application of derivatives and antiderivatives to position, velocity, displacement, and distance traveled.

In calculus I, there was a section on the application of derivatives where we discussed that the derivative of your position function yields a velocity function, and the derivative of velocity yields an acceleration function. Thus, to apply the concepts with integration, we invert the process and take antiderivatives. The antiderivative of our acceleration function yields a form for the velocity function, and the antiderivative of the velocity function yields a form for the position function.

We define position as $s(t)$, and its value corresponds to the present location of the object at time t, as measured from the origin of our coordinate plane. In short, the distance from 0.

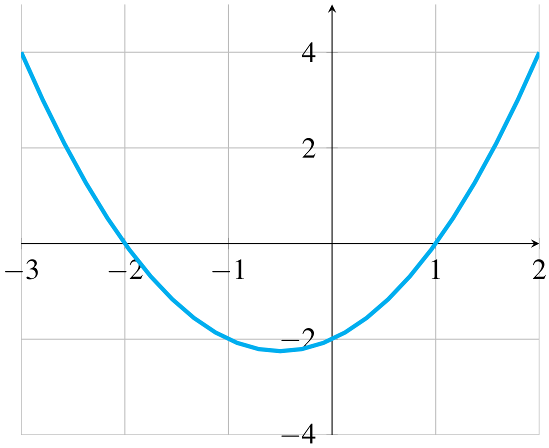
We define velocity as $v(t) = s'(t)$. This tells us the rate of change in position with respect to time.

Using this relationship, we can define the displacement of an object as the difference between its position at two times, $s(b) - s(a)$. If we only have the velocity function for the object, we can still determine its displacement through a definite integral, $\int_a^b v(t)dt = s(b) - s(a)$, by the Fundamental Theorem of Calculus.

However, displacement really only tells us about the difference between two endpoints, and does not tell us anything about the behavior of the object between those two points. We define distance traveled to be the actual distance traversed between two times. In order to do this, we need to incorporate all movement through an absolute value of the velocity function. Recall: if the velocity is negative, then the object is moving backward rather than forward. By backtracking, the displacement will be reduced. However, it is still traversing some distance during that time. So, we evaluate $\int_a^b |v(t)|dt$ in order to measure the forward and backward motion of the object. This means that we need to algebraically solve for all the times when $v(t) = 0$, and determine when the velocity is negative. We have to manually force the negative values to be positive, so we can add up all distances traveled.

■ **Example 3.1** Given the velocity function $v(t) = t^2 + t - 2$ on the interval $-3 \leq t \leq 2$

1. Graph the velocity function



2. Find the displacement from $t = -3$ to $t = 2$
$\int_{-3}^{2} \left(t^2 + t - 2\right) dt = \frac{t^3}{3} + \frac{t^2}{2} - 2t|_{-3}^{2} = \left(\frac{8}{3} + 2 - 4\right) - \left(-9 + \frac{9}{2} + 6\right)$ *units*
3. Find the distance traveled from $t = -3$ to $t = 2$
Based off the graph (or algebraically), the velocity function, $v(t) = 0$ at $t = -2$ and $t = 1$. $v(t)$ is negative between these two points, and positive outside of them. So, to determine the distance traveled, we divide up the interval at these values and use our properties of definite integrals to define $\int_{-3}^{2} |t^2 + t - 2| dt = \int_{-3}^{-2} \left(t^2 + t - 2\right) dt - \int_{-2}^{1} \left(t^2 + t - 2\right) dt + \int_{1}^{2} \left(t^2 + t - 2\right) dt$. The middle term is negative to make its output positive, so that the distance traveled is added instead of subtracted.
$\int_{-3}^{-2} \left(t^2 + t - 2\right) dt - \int_{-2}^{1} \left(t^2 + t - 2\right) dt + \int_{1}^{2} \left(t^2 + t - 2\right) dt = \left(\frac{-8}{3} + \frac{4}{2} - 2(-2)\right) - \left(\frac{-27}{3} + \frac{9}{2} - 2(-3)\right) - \left[\left(\frac{1}{3} + \frac{1}{2} - 2\right) - \left(\frac{-8}{3} + \frac{4}{2} - 2(2)\right)\right] + \left(\frac{8}{3} + \frac{4}{2} - 2(2)\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{49}{6}$
4. Find the function for position, $s(t)$, if $s(0) = 2$

   Since $s(t) = \int v(t) dt = \frac{t^3}{3} + \frac{t^2}{2} - 2t + C$, we can use the function value given to determine the value of C.

   $s(0) = \frac{0^3}{3} + \frac{0^2}{2} - 2(0) + C = C$, and the condition given said $s(0) = 2$, so $C = 2$.

   Therefore, $s(t) = \frac{t^3}{3} + \frac{t^2}{2} - 2t + 2$.

■

Much like the area functions we discussed in the previous section, we can define the function generally $s(t) = s(0) + \int_0^t v(x) dx$ as an alternative to solving algebraically for C. This form gives you the function directly.

We define acceleration as the rate of change in the velocity, so $a(t) = v'(t) = s''(t)$. We can use the same relationships to define velocity $v(t)$ and position $s(t)$ from a given acceleration function.

$v(t) = v(0) + \int_0^t a(x)dx$, which is the same form as our definition above to define the position function. Both require a given initial value of our function.

■ **Example 3.2** Determine the velocity and position functions for a ball dropped off a cliff, given constant acceleration due to gravity $a(t) = -9.8m/s^2$ if the height of the cliff is $100m$, and the ball has no initial velocity.

The description tells us that $a(t) = -9.8$, $v(0) = 0$, and $s(0) = 100$. So, we can use the equations above to solve for $v(t)$ and $s(t)$.

$v(t) = v(0) + \int_0^t -9.8dx = 0 - 9.8x|_0^t = -9.8tm/s$

$s(t) = s(0) + \int_0^t -9.8xdx = 100 - \frac{9.8x^2}{2}|_0^t = 100 - 4.9t^2m$

1. How far does the ball fall in the first $10seconds$?

   Since our velocity function is negative on the interval $0 \le t \le 10$, we know that the distance traveled will be equal to the displacement (when $v(t)$ does not change sign, this is true). So, the distance it falls is $s(10) - s(0) = (100 - 490) - (100 - 0) = -490m$. The negative implies that it is falling downward, and the distance it falls is $490m$.
2. When will the ball hit the ground? (What is a natural choice for the location of the ground? Height = 0).

   $s(t) = 0$ when $100 - 4.9t^2 = 0$, which when we solve for t algebraically yields
   $4.9t^2 = 100$,
   $t^2 = \frac{100}{4.9}$,
   $t = \pm\sqrt{\frac{100}{4.9}}$,
   However, a negative time does not make sense physically, so we only use the positive value. The ball will hit the ground at $t = \frac{10}{\sqrt{4.9}} seconds$.

■

3.2 Regions Between Curves

Previously, we used integration to define the area "under" a curve. We are extending the concept so we can find any general region between curves. When we found the area under a curve, it was always in reference to the x-axis, or $y = 0$. So, we were effectively finding the area formed by the integral $\int_a^b (f(x) - 0)\,dx$.

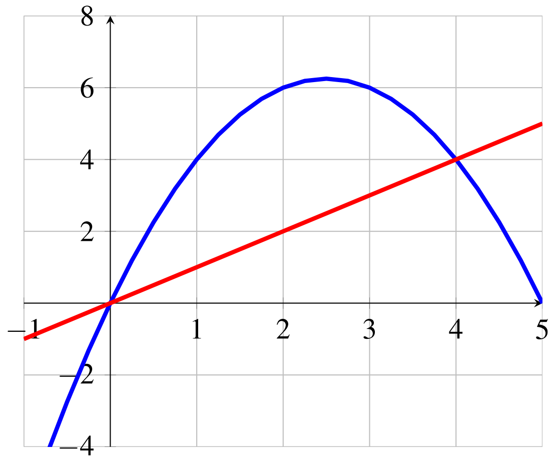
When we extend this to find the area between curves, we subtract the bottom curve from the top curve. So, if $f(x)$ is above $g(x)$ on $a \le x \le b$, we integrate $\int_a^b (f(x) - g(x))\,dx$ to define the area between the functions $f(x)$ and $g(x)$.

■ **Example 3.3** Find the area between $f(x) = 5x - x^2$ and $g(x) = x$.

Since no interval was given, we need to find the values where these two functions intersect. Set $f(x) = g(x)$.

$5x - x^2 = x$
$4x = x^2$
$x = 0, 4$

So our interval is $0 \le x \le 4$. Then, we need to determine which function is above the other on this interval. You can do this by graphing the two functions, or by algebraically substituting a value from this interval into the functions.



Algebraically, we just need to compare the function values. So, if we arbitrarily pick $x = 1$ because it is a simple number to use between 0 and 4 we get $f(1) = 4$, and $g(1) = 1$. Since $1 < 4$, we know that $f(x)$ is above $g(x)$.

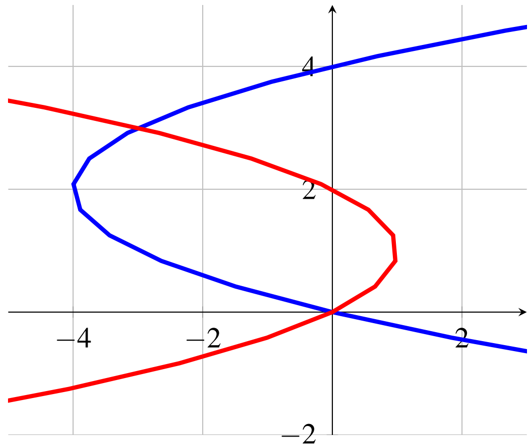
Thus, to determine the area between these two functions, we evaluate $\int_0^4 \left(5x - x^2 - x\right) dx = \frac{5x^2}{2} - \frac{x^3}{3} - \frac{x^2}{2}|_0^4 = 2x^2 - \frac{x^3}{3}|_0^4 = 32 - \frac{64}{3} - 0 + 0 = \frac{32}{3} units^2$. ■

3.2.1 Rotated Areas Between Curves

We have some sense of what to do to calculate the areas between curves when the functions are defined in terms of x. However, how do we deal with this if our functions are not nicely defined in terms of x? If we can write the functions nicely in terms of y, we can do the same process, but integrate along y instead of x. Now, instead of top - bottom, we will think of it as right - left.

■ **Example 3.4** Find the area between $f(y) = y^2 - 4y$ and $g(y) = 2y - y^2$.

If we graph these functions, their intersection is not well-defined in terms of x, and neither are the functions. We effectively rotate the problem so that our x and y axes are flipped.



The process is the same, we start by finding the intersection points $y^2-4y=2y-y^2$
$2y^2=6y$
$y^2=3y$
$y=0,3$

Then we see that $g(y)$ is on the right side of the region, and $f(y)$ is on the left side of the region, so we will integrate $g(y)-f(y)$ on the interval from $0\leq y\leq 3$.

$$\int_0^3 \left(2y-y^2-(y^2-4y)\right)dy=\int_0^3 \left(6y-2y^2\right)dy=3y^2-\frac{2y^3}{3}|_0^3=27-18-(0-0)=9units^2$$
■

■ **Example 3.5** If we were to attempt to integrate this same region along x (not recommended), we would have to solve for y:

$x=y^2-4y$, to solve for y we need to complete the square: $x=(y-2)^2-4$, and then we can solve for $y=2\pm\sqrt{x+4}$ in order to write the function in terms of x.

$x=2y-y^2$, we again need to complete the square: $x=-(y-1)^2+1$, and then $y=1\pm\sqrt{1-x}$. Notice that both of these equations are messy compared to what we did in the previous example.

Now, based on the graph from our previous example, there are actually three different regions formed because the "top" function and "bottom" function change throughout the interval.

We define the intervals using the intersection points and the vertices of the parabolas. So, the first one uses the two sides of the parabola formed by $f(y)$ on $-4\leq x\leq -3$ (which we determine by substituting $y=3$ into our equation). The second interval has the top half of $g(y)$ as the "top" function, and the bottom half of $f(y)$ as the "bottom", along $-3\leq x\leq 0$. The third interval uses the top and bottom halves of $g(y)$ along $0\leq x\leq 1$.

In terms of the integrals, this looks like:
$\int_{-4}^{-3}\left(2+\sqrt{x+4}-(2-\sqrt{x+4})\right)dx+\int_{-3}^{0}\left(1+\sqrt{1-x}-(2-\sqrt{x+4})\right)dx+\int_0^1\left(1+\sqrt{1-x}-(1-\sqrt{1-x})\right)dx$
This is solvable, but as you can tell - much more work. ■

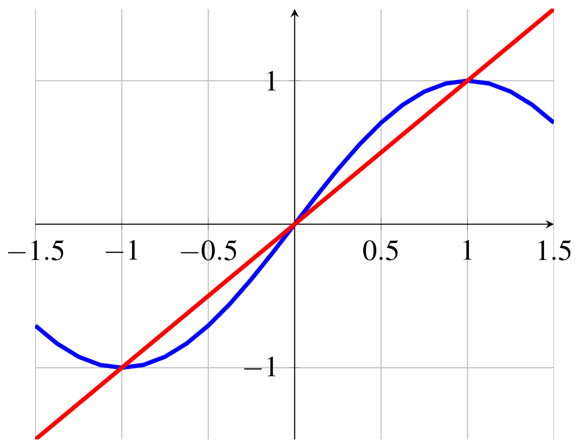
3.2.2 Crossing Functions

When we have two functions that cross more than twice, the areas formed also need to be split up in a similar way to the previous example.

■ **Example 3.6** Determine the area formed between $f(x) = \sin\left(\frac{\pi x}{2}\right)$ and $g(x) = x$.

First, when does $\sin\left(\frac{\pi x}{2}\right) = x$?
We know that $\sin(0) = 0$ and $x = 0$ will pass through the same point.
We also know that $\sin\left(\frac{\pi}{2}\right) = 1$, which happens when $x = 1$, so when $x = 1$, they will also pass through the same point.
Finally, we know that $\sin\left(\frac{-\pi}{2}\right) = -1$, which happens when $x = -1$, so when $x = -1$, they will also pass through the same point.

Graphically, we can see there are only these three intersections.



We can also note that $g(x)$ is greater than $f(x)$ on $-1 \leq x \leq 0$, and $f(x)$ is greater than $g(x)$ on $0 \leq x \leq 1$.

Thus, the area formed is defined by $\int_{-1}^{0} (g(x) - f(x))\, dx + \int_{0}^{1} (f(x) - g(x))\, dx$
$\int_{-1}^{0} \left(x - \sin\left(\frac{\pi x}{2}\right)\right) dx + \int_{0}^{1} \left(\sin\left(\frac{\pi x}{2}\right) - x\right) dx$
$= \frac{x^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) |_{-1}^{0} + \frac{-2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}|_0^1$
$= \left[\left(0 + \frac{2}{\pi}\right) - \left(\frac{1}{2} + 0\right)\right] + \left[\left(0 - \frac{1}{2}\right) - \left(\frac{-2}{\pi} - 0\right)\right]$
$= \frac{2}{\pi} - \frac{1}{2} - \frac{1}{2} + \frac{2}{\pi} = \frac{4}{\pi} - 1 units^2$ ■

In summary, if you are integrating along x, you will take the difference " Top function - Bottom function" and integrate along the x interval.

If you are integrating along y, you will take the difference " Right function - Left function" and integrate along the y interval.

If the "Top function" becomes the "Bottom function", vice versa, or there are additional intersections between the two functions -> Break up the interval!

3.3 Volumes by Slicing

This is an extension of the area between curves, because we take an area between curves and then rotate it about an axis to form a three dimensional volume of rotation. Because the entire area is rotated through space, the rotation itself forms circles about that axis, and each of these circles has an area πr^2.

When we compute volumes by slicing, we add up all these circles, so along the x-axis this is $\pi r^2 \delta x$

■ **Example 3.7** Determine the volume of a cone formed by $y = 2x$ between $x = 0$ and $x = 1$, rotated about the x-axis.

Note that every point on the line $y = 2x$ is fixed at its x position, and the y value is a fixed distance from the x-axis as it rotates. Thus, $r = y = 2x$, so our area function $A = \pi(2x)^2$, which we will integrate from 0 to 1.

$\int_0^1 \pi 4x^2 dx = \frac{4\pi x^3}{3}|_0^1 = \frac{4\pi}{3} - 0 = \frac{4\pi}{3} units^3$ ■

3.3.1 Disk Method

This is often referred to as the “Disk method" because the rotation of each piece forms a disk. Note that this is only for the cases where we have the area measured with respect to the axis of rotation.

For this method, we always integrate with respect to the same variable as the axis of rotation.

Rotating about the x-axis: $\int_a^b A(x)dx$

Rotating about the y-axis: $\int_c^d A(y)dy$

■ **Example 3.8** If we rotate the same function about the y-axis, what is the volume of rotation?

We need to convert $y = 2x$ into $x = \frac{y}{2}$, and we need to update the bounds on x $(0 \le x \le 1)$ to y. Since the points associated with these values are $(0,0)$ and $(1,2)$, the new bounds are $0 \le y \le 2$.

$A(y) = \pi x^2 = \pi \left(\frac{y}{2}\right)^2 = \frac{\pi y^2}{4}$

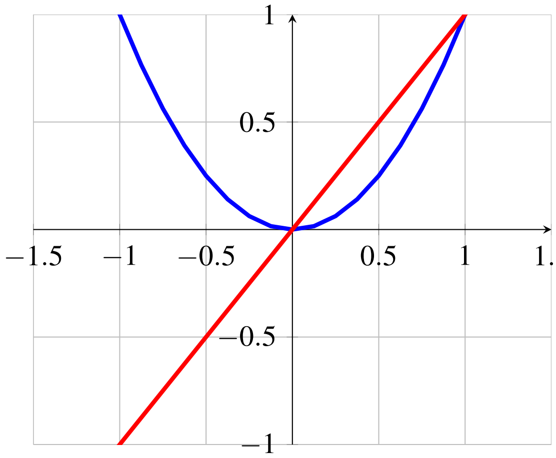
$\int_0^2 \frac{\pi y^2}{4} dy = \frac{\pi y^3}{12}|_0^2 = \frac{\pi 2^3}{12} - 0 = \frac{2\pi}{3} units^3$

Concept check: why is this value different? The function is closer to the y-axis, so the area we are rotating is smaller, which leads to a smaller volume also. ■

3.3.2 Washer Method

The washer method does not require the area be adjacent to the axis of rotation. This method uses the area between two functions, rotated about any axis (not just the x or y axis, but also any horizontal or vertical line like $x = 1$, or $y = -2$, etc.).

■ **Example 3.9** Determine the volume formed by rotating the area between $y = x^2$ and $y = x$ about the x-axis.



As we did in the previous section, we will find the difference of two functions. However, since we are forming a volume of rotation, we will calculate the volume formed by each function, and take the difference of those two calculations. In this case, we always use the function furthest from the axis of rotation as the 'outer function' and the closer function as the 'inner function' so that when we calculate the volume it is $V_{outer} - V_{inner}$.

In this problem, the line $y = x$ is further from the x-axis and forms a cone, and the curve $y = x^2$ is basically carving a section of the cone out. So, we need the full volume of the cone first, and then we subtract the volume formed by the parabola.

$$\int_0^1 \pi x^2 dx - \int_0^1 \pi x^4 dx$$
$$\pi \frac{x^3}{3}|_0^1 - \pi \frac{x^5}{5}|_0^1$$
$$\pi \left(\frac{1}{3} - 0\right) - \pi \left(\frac{1}{5} - 0\right)$$
$$\pi \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2}{15} units^3$$

■

3.3.3 Rotating about a Line

This shifts the entire problem so that all the functions are measured by their distance from that line. Computationally, this really just shows up as an additional constant in the function.

■ **Example 3.10** Determine the volume formed by rotating the area under $y = 2x$ on $0 \leq x \leq 1$ about $y = 2$.

Now, the radius is formed by the distance between $y = 2$ and $y = 2x$ for the 'inner function', and the distance between $y = 2$ and $y = 0$ for the 'outer function'.

$$\int_0^1 \pi(2-0)^2 dx - \int_0^1 \pi(2-2x)^2 dx$$
$$\int_0^1 4\pi dx - \int_)^1 \pi(4 - 8x + 4x^2) dx$$
$$4\pi x|_0^1 - \pi(4x - 4x^2 + \frac{4x^3}{3})|_0^1$$

$4\pi(1-0)-\pi\left[(4-4+\frac{4}{3})-(0-0+0)\right]=4\pi-\frac{4\pi}{3}=\frac{8\pi}{3}units^3$

Note: this is significantly larger than our earlier answer because it is rotated about a different axis that is further from the area, so the volume will look completely different. ■

3.4 Arc Length

Arc length allows us to define the length of any curve (as though we flattened it out to a line). The form comes from our Pythagorean theorem, which we used to define the distance between two points (distance formula), or the length of a hypotenuse of a triangle.

If we use L to represent the arc length, then when we take small steps along the curve from some (x_0,y_0) to $(x_0+\Delta x, y_0+\Delta y)$, then we can define the length through $\Delta L=\sqrt{(x_0+\Delta x-x_0)^2+(y_0+\Delta y-y_0)^2}=\sqrt{(\Delta x)^2+(\Delta y)^2}$. This can be factored to be written in terms of Δx to approximate the form of an integral (as we did back in Calculus I).

$$\Delta L=\sqrt{(\Delta x)^2\left(1+\left(\frac{\Delta y}{\Delta x}\right)^2\right)}=\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$

In the limit as $\Delta x\to 0$, $\Delta x\to dx$ and $\frac{\Delta y}{\Delta x}\to\frac{dy}{dx}=f'(x)$.

This gives us the general formula for calculating the arc length of any given function $f(x)$: $L=\int_a^b\sqrt{1+(f'(x))^2}dx$

■ **Example 3.11** We will begin with a proof of concept example: a line. Since we can use the Pythagorean theorem to define the length along a line, we can verify that the technique works (at least for this function).

Determine the arc length of $f(x)=4x-3$ along $0\le x\le 3$.

First, we find $f'(x)=4$

Then, we substitute the derivative into our form for the arc length: $L=\int_0^3\sqrt{1+(4)^2}dx=\int_0^3\sqrt{17}dx=\sqrt{17}x|_0^3=\sqrt{17}(3-0)=3\sqrt{17}units$

Comparison with Pythagorean theorem: the endpoints are $(0,-3)$ and $(3,9)$, so the change in x is $a=3$, and the change in y is $b=12$. $a^2+b^2=c^2$, $9+144=153=9*17$, so $c=\sqrt{9*17}=3\sqrt{17}units$ ✓ ■

■ **Example 3.12** Determine the arc length of $f(x)=2x^{3/2}$ along $1\le x\le 4$

First, find $f'(x)=2*\frac{3}{2}x^{1/2}$

Then, set up the integral $\int_1^4\sqrt{1+(3\sqrt{x})^2}dx=\int_1^4\sqrt{1+9x}dx$

In order to integrate this, we apply a u-substitution: $u=1+9x$, $\frac{du}{dx}=9$, so we replace $dx=\frac{du}{9}$ and update the bounds. When $x=1$, $u=1+9=10$, and when $x=4$, $u=1+36=37$.

$$\int_{10}^{37} \sqrt{u}\frac{du}{9} = \frac{1}{9}\left(\frac{2u^{3/2}}{3}\right)\Big|_{10}^{37} = \frac{2}{27}\left(37^{3/2} - 10^{3/2}\right) units$$ ■

3.5 Specific Physical Problems

3.5.1 Density and Mass

Recall: Mass = Density×Volume (3D)

If we constrain this to 1D, then the density function must be with respect to length instead of volume so that Density $= \frac{\text{Mass}}{\text{Length}}$. If Mass $= m$, Density $= \rho(x)$, and Length is Δx, then $m = \int_a^b \rho(x)dx$

■ **Example 3.13** Given a thin bar with density function $\rho(x) = 2 - \frac{x}{2} kg/m$, and it is $2m$ long, what is the mass of the bar?

$$m = \int_0^2 \left(2 - \frac{x}{2}\right) dx = 2x - \frac{x^2}{4}\Big|_0^2 = (4-1) - (0-0) = 3kg$$ ■

3.5.2 Work

Work is done by a force acting along a distance, $F \cdot \Delta x$.

In an integral form, we can determine the work done by moving an object through $W = \int_a^b F(x)dx$

■ **Example 3.14** How much work is done by moving an object experiencing a magnetic field force, $F(x) = \frac{2}{x^2}N$, from $x = 1m$ to $x = 3m$?

$$W = \int_a^b F(x)dx = \int_1^3 \frac{2}{x^2}dx = \frac{-2}{x}\Big|_1^3 = \left(\frac{-2}{3} - \left(\frac{-2}{1}\right)\right) = \frac{4}{3}Nm \text{ (or } J\text{)}$$

■

A specific application of work is Hooke's Law for spring systems. In these problems, the forcing function is defined through the spring constant, k. $F(x) = kx$, where positive x corresponds to stretching the spring, and negative x corresponds to compression of the spring from equilibrium.

■ **Example 3.15** For a spring with $k = 50N/m$, how much work will it take to stretch the spring $\frac{1}{2}m$ from equilibrium?

$$W = \int_a^b F(x)dx = \int_0^{1/2} 50xdx = 25x^2\Big|_0^{1/2} = 25\left(\frac{1}{4} - 0\right) = \frac{25}{4}Nm \text{ (or } J\text{)}$$

For the same spring, how much work would it take to compress it $\frac{1}{5}m$?

$$W = \int_a^b F(x)dx = \int_0^{-1/5} 50xdx = 25x^2\Big|_0^{-1/5} = 25\left(\frac{1}{25} - 0\right) = 1Nm \text{ (or } J\text{)}$$

■

3.5.3 Lifting Problems

To lift a solid object, we can use the same relationship $W = F \cdot d$ from earlier. For a fluid this is more complicated, but the underlying principles are the same.

■ **Example 3.16** Pumping water out of a bucket: We have a cylindrical bucket with radius $0.5m$ and height $1m$. Assuming it is filled to the top with water, how much work will be done to pump water up to the top and out of the bucket? Assume that the water is pumped from the top of the current water level as it is decreasing.

The general form of the equation for these types of problems is: $W = \int_a^b \rho g A(y) D(y) dy$

ρ is the density of the fluid
g is the acceleration due to gravity
$A(y)$ is an area function defining the cross-section that is the surface of the water at height y
$D(y)$ is the distance from the top of the water to the top of the bucket

$\rho_{water} = 1000 kg/m^3$
$g = 9.8 m/s^2$
$A(y) = \pi r^2 m^2$ (area of a circle)
$D(y) = (h-y)m$

$$W = \int_0^h (1000[kg/m^3])(9.8[kg/m^2])(\pi r^2[m^2])(h-y)[m]dy[m]$$
$$= \int_0^1 1000(9.8)\left(\frac{\pi}{4}\right)(1-y)dy[kgm^2/s^2]$$
$$= \frac{9800\pi}{4}\int_0^1 (1-y)dy[Nm]$$
$$= \frac{9800\pi}{4}\left(y-\frac{y^2}{2}\right)\Big|_0^1[Nm] = \frac{9800\pi}{4}\left(\left(1-\frac{1}{2}\right)-(0-0)\right)Nm = \frac{9800\pi}{8}Nm = 1225\pi Nm$$

If the same bucket started only half-full, how much work will it take to pump all that water out? Does it match your hypothesis?

$$W = \int_0^h (1000[kg/m^3])(9.8[kg/m^2])(\pi r^2[m^2])(h-y)[m]dy[m]$$
$$= \int_0^{1/2} 1000(9.8)\left(\frac{\pi}{4}\right)(1-y)dy[kgm^2/s^2]$$
$$= \frac{9800\pi}{4}\int_0^{1/2} (1-y)dy[Nm]$$
$$= \frac{9800\pi}{4}\left(y-\frac{y^2}{2}\right)\Big|_0^{1/2}[Nm] = \frac{9800\pi}{4}\left(\left(\frac{1}{2}-\frac{1}{8}\right)-(0-0)\right)Nm = \frac{9800\pi}{4}\frac{3}{8}Nm = \frac{3}{4}(1225)\pi Nm$$

This is noteworthy because it takes $\frac{3}{4}$ the total work to pump out only the lower half of the bucket's water. The further the distance traveled to the top, the more work is required. ■

3.5.4 Force and Pressure Problems

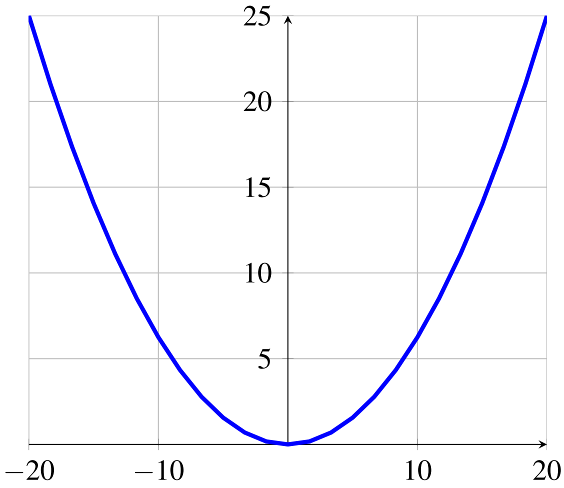
This section is focused on hydrostatic pressure (same on bottom and sides). This is determined using the force due to the weight of water, $F = mg = \rho V g$, where $m =$ mass, $g =$ acceleration due to gravity, $\rho =$ density, and $V =$ volume. Since we are not truly working in 3D, we will assume symmetry in the volume so that we can define it through $V = Ah$, where $A =$ cross-sectional area.

Pressure is defined through $P = \frac{F}{A} = \frac{\rho A h g}{A} = \rho h g$ in this case.

Force on the bottom due to hydrostatic pressure is then determined through $F = \int_0^h \rho g(h-y)w(y)dy$, where $(h-y)$ is the depth, and $w(y)dy$ defines the cross-sectional area for our volume.

■ **Example 3.17** The lower edge of a dam is defined by the parabola $y = \frac{x^2}{16}$. Determine the force on the dam if its height is $25m$.

First, draw the shape of the dam for reference:



Notice that the width of this shape at each height is given by $2\times x$, and because the function defining the dam is $y=\frac{x^2}{16}$, we can solve this equation for $x=4\sqrt{y}$ and define $w(y)=8\sqrt{y}$ (the width of the dam at each value of y).

Then, we can use the form $F=\int_0^h \rho g(h-y)w(y)dy$
$=\int_0^{25}(1000[kg/m^3])(9.8[m/s^2])(25-y)[m](8\sqrt{y}[m])dy[m]$
$=98008\dot{}\int_0^{25}(25\sqrt{y}-y^{3/2})dy[kgm/s^2]$
$=98008\dot{}\left(\frac{50y^{3/2}}{3}-\frac{2y^{5/2}}{5}\right)|_0^{25}[N]$
$=98008\dot{}\left(\frac{50\dot{1}25}{3}-\frac{2\dot{5}^5}{5}\right)N=\frac{196000000}{3}N$ ■

4. Logarithmic, Exponential, and Hyperbolic Functi

4.1 Logarithmic and Exponential Functions

In this section, we use properties of logarithmic and exponential functions to evaluate integrals involving them.

4.1.1 Logarithmic Functions

Recall from Calculus I:
The general antiderivative $\int \frac{1}{x} dx = \ln|x| + C$, where $\ln x = \int_1^x \frac{1}{t} dt$ (this assumes only positive values of x), and $\ln(1) = 0$.
$\frac{d}{dx}(\ln x) = \frac{1}{x}$

Recall from Algebra:
Domain of $\ln x$ is $(0, \infty)$
Range of $\ln x$ is $(-\infty, \infty)$.
Logarithm rules:
$\ln(xy) = \ln(x) + \ln(y)$
$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
$\ln(x^p) = p\ln(x)$

We will use these to determine integrals of logarithmic functions.

■ **Example 4.1** Evaluate $\int_3^4 \frac{x}{x^2-4} dx$ and simplify as much as possible using logarithm rules.
In order to integrate, we must apply a u-substitution: $u = x^2 - 4$, so $du = 2xdx$
Update bounds: when $x = 3$, $u = 3^2 - 4 = 5$, and when $x = 4$, $u = 4^2 - 4 = 12$.
Updated integral: $\frac{1}{2}\int_5^{12} \frac{du}{u} = \frac{1}{2}\ln|u||_5^{12} = \frac{1}{2}(\ln(12) - \ln(5)) = \frac{1}{2}\ln\left(\frac{12}{5}\right) = \ln\left(\sqrt{\frac{12}{5}}\right)$ ■

4.1.2 Exponential Functions

Recall from Algebra: The functions $\ln(x)$ and e^x are inverses of each other (they "undo" each others' operations). So, $e^{\ln(x)} = x$, and $\ln(e^x) = x$.
Exponential base $e \approx 2.71828$ (we will approximate it as a little less than 3)
Exponential Properties:
$e^x e^y = e^{x+y}$
$\frac{e^x}{e^y} = e^{x-y}$
$(e^x)^y = e^{xy}$

Recall from Calculus I:
$\frac{d}{dx}(e^x) = e^x$
$\int e^x dx = e^x + C$

Use these properties to evaluate integrals.

Example 4.2 Integrate $\int xe^{x^2} dx$
In order to solve, we need to apply a u-substitution: $u = x^2$, $du = 2xdx$
Updated integral: $\int \frac{1}{2} e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$ ■

4.1.3 General Bases of Logarithmic and Exponential Functions

We denote general bases by b, where a general base exponential is b^x, and a general base logarithm is $\log_b(x)$.
We can define each of these in relation to our natural base e from earlier: $b^x = e^{x \ln b}$, and $\log_b(x) = \frac{\ln x}{\ln b}$.
Using these relations, we can define their derivatives using chain rule: $\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x \ln b}\right) = e^{x \ln b} \ln b = b^x \ln b$
$\frac{d}{dx}(\log_b(x)) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{x} \frac{1}{\ln b}$
We can also define the integral of b^x using u-substitution: $\int b^x dx = \int e^{x \ln b} dx$
$u = x \ln b$, $dx = dx \ln b$
Updated integral $\int \frac{1}{\ln b} e^u du = \frac{1}{\ln b} e^u + C = \frac{1}{\ln b} e^{x \ln b} + C = \frac{b^x}{\ln b} + C$

Example 4.3 Integrate $\int 5^{3x} dx$
There are multiple ways we can approach this problem. Lets approach it with a u-substitution and the properties we defined above.
$u = 3x$, $du = 3dx$
Updated integral: $\frac{1}{3} \int 5^u du = \frac{1}{3} \frac{5^u}{\ln 5} + C = \frac{1}{3} \frac{5^{3x}}{\ln 5} + C$ ■

4.1.4 Generalized Power Rule

We can use the relationships between logarithmic and exponential to define a generalized power rule using $x^p = e^{p \ln x}$.

Example 4.4 Determine the derivative of x^{3x} using the exponential form of the power function.
Rewrite $x^{3x} = e^{3x \ln x}$, and define the derivative using the chain rule.

$$\frac{d}{dx}\left(e^{3x\ln x}\right) = e^{3x\ln x}\left(3\ln x + \frac{3x}{x}\right) = x^{3x}\left(3\ln x + 3\right)$$

Note: this requires a chain rule with a product rule to arrive at the correct answer. ■

4.2 Exponential Models

Now that we've reviewed and applied properties of Logarithmic and Exponential functions, we can discuss specific applications of these functions.

4.2.1 Exponential Growth and Decay

Functions modeling exponential growth and decay are of the form $y(t) = Ce^{kt}$, where $k =$ rate constant (growth or decay), and $C =$ initial value (initial population).

Note that $\frac{dy}{dt} = Cke^{kt} = ky$, the rate of change in the population is directly proportional to the current population and the rate constant.

When there is exponential growth, $k > 0$, and y will increase as t increases. A specific interest in such problems is the time it takes to double the population. If C is the initial population, then we seek the time when $y = 2C$. We can solve for this algebraically:
$2C = Ce^{kt}$, so $2 = e^{kt}$. In order to isolate t to determine the doubling time, we apply a natural logarithm to both sides of the equation, $\ln 2 = kt$, so $t = \frac{\ln 2}{k}$. This will be true for any exponential growth case of the form $y(t) = Ce^{kt}$.

When there is exponential decay, $k < 0$, and y will decrease as t increases. A specific interest in such problems is the time it takes to halve the population (or in radioactive decay, halve the mass of material). If C is the initial population, then we seek the time when $y = \frac{1}{2}C$. We can solve for this similarly using algebra:
$\frac{1}{2}C = Ce^{kt}$, so $\frac{1}{2} = e^{kt}$. In order to isolate t, we apply a natural logarithm to both sides of the equation $-\ln 2 = kt$ (use your logarithm properties!), so that $t = \frac{-\ln 2}{k}$

■ **Example 4.5** Given a population of 500 bacteria with a growth rate of 0.5, define the model for its exponential growth, and determine its doubling time in hours.

$$y(t) = 500e^{0.5t}$$
$$t_{double} = \frac{\ln 2}{k} = \frac{\ln 2}{0.5}$$
■

Financial Models

Exponential models for financial applications expand upon the form of exponential models. In these cases, we focus on Annual Percentage Yield (% increase per year). Since this is defined in specific time increments, we have to determine the value of k through the exponential model, $y(t) = Ce^{kt}$.

■ **Example 4.6** Define an exponential model that increases by 5% each year.
If we assume t is in years, then $y(1) = 1.05y(0)$, substituting the form of the exponential model yields $Ce^{k} = 1.05C$, so $e^{k} = 1.05$, and $k = \ln(1.05)$.
Substituting the value of k back into the model yields: $y(t) = Ce^{\ln(1.05)t}$
Note: the form of k in these cases is $\ln(1+\%)$ ■

■ **Example 4.7** What is the balance after 5 years if you invest \$300 in an account accruing 3% APY?

$k = \ln(1+0.03) = \ln(1.03)$
$y(t) = 300e^{\ln(1.03)t}$
After 5 years, $y(5) = 300e^{\ln(1.03)5}$

What is the doubling time for your investment at this APY?

$600 = 300e^{\ln(1.03)t}$, $t = \dfrac{\ln 2}{\ln(1.03)}$ years. ■

Energy Consumption

Energy consumption models have an annual % increase, and are treated exactly the same as APY in the last problems.

■ **Example 4.8** If the initial consumption is 5000 W, and there is a 2% increase in consumption each year, define a model for the power consumed.

A 2% increase per year results in $k = \ln(1.02)$, so $P(t) = 5000e^{\ln(1.02)t}$.

What is the total energy usage in the 2nd year?
Since $P(t)$ defines the power usage at each time t, in order to find the total energy usage, we need to add up all the energy usage between years 1 and 2 → Integrate!
$\int_1^2 5000e^{\ln(1.02)t}dt = \dfrac{5000}{\ln(1.02)} e^{\ln(1.02)t}|_1^2 =$
$\dfrac{5000}{\ln(1.02)}\left(e^{\ln(1.02)2} - e^{\ln(1.02)}\right) = \dfrac{5000}{\ln(1.02)}\left((1.02)^2 - 1.02\right) W\cdot$years. ■

Pharmacokinetics

Pharmacokinetics model drug levels in the bloodstream, and are generally defined by the half-life of the drug.
Given the half-life of a drug, define the model $y(t) = Ce^{kt}$, where k is the rate of decrease per hour.

■ **Example 4.9** If you take 200mg of Ibuprofen, and it has a half-life of 4 hours: define the exponential model

$y(t) = 200e^{kt}$, since the half-life is 4 hours, $y(4) = 100 = 200e^{4k}$.

Solving for k, we have $\dfrac{1}{2} = e^{4k}$, so $-\ln(2) = 4k$, and $k = \dfrac{-\ln(2)}{4}$.
$y(t) = 200e^{-\ln(2)t/4}$

How long until the level reaches 5% of the initial dose?

$0.05 * 200 = 200e^{-\ln(2)t/4}$
$0.05 = e^{-\ln(2)t/4}$
$\ln(0.05) = \dfrac{-\ln(2)t}{4}$
$\dfrac{4\ln(0.05)}{-\ln(2)} = t \approx 17.2877$ hours.

When will the level reach 40mg?

$$40 = 200e^{-\ln(2)t/4}$$
$$0.2 = e^{-\ln(2)t/4}$$
$$\ln(0.2) = \frac{-\ln(2)t}{4}$$
$$\frac{4\ln(0.2)}{-\ln(2)} = t \approx 9.2877 \text{ hours.}$$

■

4.3 Hyperbolic Functions

Hyperbolic functions are defined through a combination of exponential functions. These are really nice functions to work with, that arise in some applications. All of them can arise later, but we will focus mostly on $\cosh(x)$ and $\sinh(x)$.

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

The remaining hyperbolic functions are defined in the same manner as the trigonometric functions.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$
$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

4.3.1 Graphs

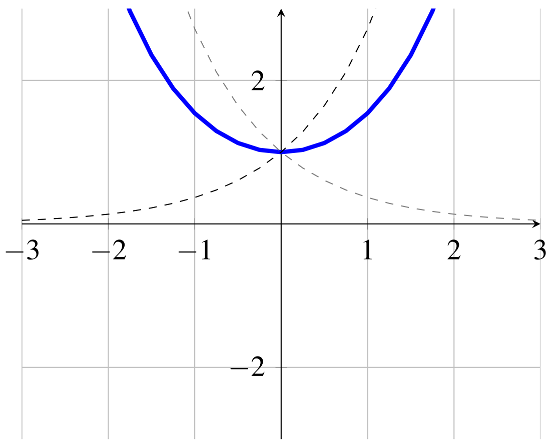


Figure 4.1: Above is the graph of $cosh(x)$ with the solid curve, and included are the graphs of e^x and e^{-x} which are used to define the function $cosh(x)$ so you can see how they compare to each other.

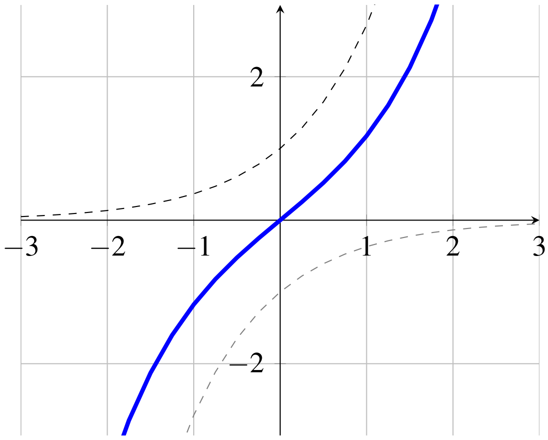


Figure 4.2: Above is the graph of $\sinh(x)$ with the solid curve, and included are the graphs of e^x and $-e^{-x}$ which are used to define the function $\sinh(x)$ so you can see how they compare to each other.

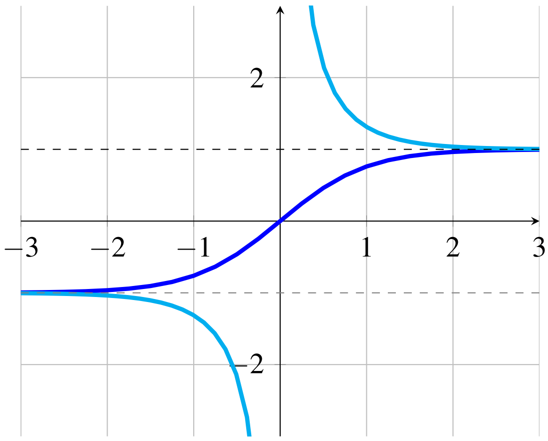


Figure 4.3: Above shows the graphs of $\tanh(x)$ with the blue solid curve, and $\coth(x)$ with the cyan solid curve. Included are the horizontal asymptotes at $y = 1$ and $y = -1$ for reference.

We include plots of all the functions here for reference, however the ones you need to be the most familiar with are $\cosh(x)$ and $\sinh(x)$.

4.3.2 Identities

Note that $\cosh(x)$ is an even function which is shown by its graph because it is symmetric about the y-axis, $\cosh(-x) = \cosh(x)$ (just as $\cos(x)$ is also an even function). Similarly, $\sinh(x)$ is an odd function which is shown by its graph because it is symmetric about the origin, $\sinh(-x) = -\sinh(x)$ (just as $\sin(x)$ is also an odd function).

Pythagorean identities for hyperbolic functions:
$\cosh^2(x) - \sinh^2(x) = 1$
$1 - \tanh^2(x) = \text{sech}^2(x)$
$\coth^2(x) - 1 = \text{csch}^2(x)$

Sum identities:

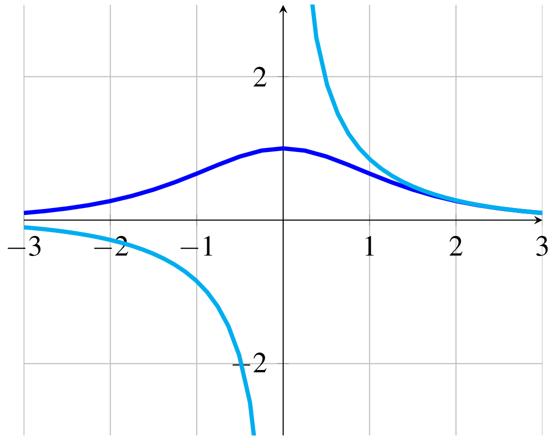


Figure 4.4: Above shows the graphs of sech(x) with the blue solid curve, and csch(x) with the cyan solid curve.

$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

Double-value identities:
$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$
$\sinh(2x) = 2\sinh(x)\cosh(x)$

Half-value identities:
$\cosh^2(x) = \dfrac{\cosh(2x)+1}{2}$
$\sinh^2(x) = \dfrac{\cosh(2x)-1}{2}$

Note that there are similarities to the same identities you used for trigonometric functions - but they are different.

4.3.3 Derivatives and Integrals

Hyperbolic functions are particularly nice for derivatives and integrals.

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$
$$\frac{d}{dx}(\sinh(x)) = \cosh(x)$$

Note: there is NO sign change. So, these are much nicer to work with than their trigonometric counterparts. You will need to know the derivatives and antiderivatives for $\cosh(x)$ and $\sinh(x)$.

$$\int \cosh(x)dx = \sinh(x) + C$$
$$\int \sinh(x)dx = \cosh(x) + C$$

■ **Example 4.10** Evaluate $\int_0^1 \cosh^3(x)\sinh(x)dx$

We notice that the derivative of $\cosh(x)$ is $\sinh(x)$ and so we can apply a u-substitution

$u = \cosh(x)$, $du = \sinh(x)dx$. The bounds are then updated, when $x = 0$, $\cosh(x) = 1$, and when $x = 1$, $\cosh(x) = \cosh(1)$ (we can define this through e^x, but it is not necessary).

$$\int_1^{\cosh(1)} u^3 du = \frac{u^4}{4}\Big|_1^{\cosh(1)} = \frac{1}{4}\left((\cosh(1))^4 - 1^4\right)$$

■

The derivatives and integrals of the remaining functions can be determined through appropriate derivative and integral rules, but here is a list of them for reference: (I recommend checking the derivatives yourself! They are good review practice before we start the next chapter!)

$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x)$

$\frac{d}{dx}(\coth(x)) = -\operatorname{csch}^2(x)$

$\frac{d}{dx}(\operatorname{sech}(x)) = -\operatorname{sech}(x)\tanh(x)$

$\frac{d}{dx}(\operatorname{csch}(x)) = -\operatorname{csch}(x)\coth(x)$

$\int \tanh(x)dx = \ln(\cosh(x)) + C$ (note that $\cosh(x)$ is always ≥ 1, so the absolute value is not necessary.)

$\int \coth(x)dx = \ln|\sinh(x)| + C$

$\int \operatorname{sech}(x)dx = \arctan(\sinh(x)) + C$

$\int \operatorname{csch}(x)dx = \ln|\tanh(x/2)| + C$

4.3.4 Inverse Hyperbolic Functions

Our hyperbolic functions, $\cosh(x)$ and $\sinh(x)$ (as well as the others), are all defined through e^x. Thus, their inverse functions are defined through $\ln(x)$. (Note: DO NOT memorize these, they are for reference.)

$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$

$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$

Derivatives:

$\frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{x^2 - 1}}$, for $x \geq 1$

$\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{x^2 + 1}}$

Integrals that yield inverse hyperbolic functions:

$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$, for $x > a$

$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$, for all x

5. Integration Techniques

5.1 First Approaches

The techniques in this section give you ways to manipulate the form of an integral so you can integrate it directly.

Recall: u-substitution

■ **Example 5.1** Evaluate $\int_1^{e^2} \frac{(\ln(x))^3}{x} dx$
We cannot simply integrate this as-is, so we apply a u-substitution to rewrite it in a form that we can integrate directly. $u = \ln(x)$, $du = \frac{1}{x} dx$, when $x = 1$, $\ln(x) = 0$, and when $x = e^2$, $\ln(x) = 2$.
$\int_0^2 u^3 du$, this integral we CAN integrate directly because we have a power rule now.
$\frac{u^4}{4}|_0^2 = \frac{1}{4}\left(2^4 - 0^4\right) = \frac{16}{4} = 4$
■

Each of these techniques is similarly attempting to simplify the integrals in order to evaluate them.

5.1.1 Split Fractions

Recall: Fractions! And their algebraic manipulations. The important note for this technique is that you can separate the numerator of a fraction into two separate fractions (we CANNOT do this with a denominator, but more on that later).

■ **Example 5.2** Evaluate $\int \frac{\sin(t) + \tan(t)}{\cos^2(t)} dt$ (Note: we're using t instead of x because these variables are just placeholders - they are arbitrary. It is important for you to get accustomed to any variable, and not just x.)
In order to work with these functions and integrate them directly, we need to split the numerator

and work with each piece.

$\int \left(\frac{\sin(t)}{\cos^2(t)} + \frac{\tan(t)}{\cos^2(t)}\right) dt = \int \left(\sec(t)\tan(t) + \tan(t)\sec^2(t)\right) dt$

Here, the first term we can recognize as the derivative of $\sec(t)$, and in the second term we can recognize that $\sec^2(t)$ is the derivative of $\tan(t)$, so we can apply a u-substitution to that term.

Let's evaluate each one separately,

$\int \sec(t)\tan(t)dt = \sec(t) + C$

$\int \tan(t)\sec^2(t)dt$, $u = \tan(t)$, $du = \sec^2(t)dt$

$\int u du = \frac{u^2}{2} + C = \frac{1}{2}\tan^2(t) + C$

Putting the two together we have:

$\int \frac{\sin(t)+\tan(t)}{\cos^2(t)} dt = \sec(t) + \frac{1}{2}\tan^2(t) + C$ ■

5.1.2 Multiply by One

This is an algebraic simplification that allows us to rewrite rational expressions after multiplying both the numerator and denominator by the same value. After simplifying algebraically, we can integrate the result.

■ **Example 5.3** Evaluate $\int \frac{1}{x^{-1}+1} dx$

We cannot simply integrate this as-is, and a power x^{-1} is difficult to work with. However, $x \cdot x^{-1} = 1$ (so long as $x \neq 0$), so let's see what multiplying by $1 = \frac{x}{x}$ will simplify to.

$\int \frac{x}{x}\frac{1}{x^{-1}+1} dx = \int \frac{x}{1+x} dx$

Here, we cannot separate terms of a denominator, but we can separate terms of a numerator. So, if we apply a u-substitution where $u = 1+x$, then $x = u-1$, and $du = dx$.

$\int \frac{u-1}{u} du = \int \left(1 - \frac{1}{u}\right) du$

This we can integrate directly, and then map back to a function of x.

$\int \left(1 - \frac{1}{u}\right) du = u - \ln|u| + C = (1+x) - \ln|1+x| + C$

■

■ **Example 5.4** Evaluate $\int \frac{1}{1-\sin(x)} dx$

We cannot simply integrate this as-is. How can we simplify this?

The key lies in the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$, specifically the form $1 - \sin^2(x) = \cos^2(x)$.

In order to get that form, we must multiply $(1-\sin(x))$ by $(1+\sin(x))$, so our $1 = \frac{1+\sin(x)}{1+\sin(x)}$.

$\int \frac{1+\sin(x)}{1+\sin(x)}\frac{1}{1-\sin(x)} dx = \int \frac{1+\sin(x)}{1-\sin^2(x)} dx = \int \frac{1+\sin(x)}{\cos^2(x)} dx$

Now, we're starting to see functions we can integrate.

$\int \left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right) dx = \int \left(\sec^2(x) + \sec(x)\tan(x)\right) dx$

These are functions we know the antiderivatives of!

$\int \sec^2(x)dx = \tan(x) + C$, and $\int \sec(x)\tan(x)dx = \sec(x) + C$

So, $\int \frac{1}{1-\sin(x)} dx = \tan(x) + \sec(x) + C$ ■

5.1.3 Long Division

This is another algebraic simplification for rational expressions with a higher order polynomial in the numerator than that in the denominator.

■ **Example 5.5** Evaluate $\int \frac{x^2+2}{x-1}dx$
Since the order (highest power of x) of x^2+2 is 2, and the order of $x-1$ is 1, we can reduce this rational expression using long division.

$x-1\overline{)x^2+0x+2}$ Note that this form represents the same thing as $\frac{x^2+0x+2}{x-1}$
The question to ask yourself is "What do we multiply the x in $(x-1)$ by, in order to make an x^2?" Because we are trying to determine "How many" of the factors $x-1$ are in x^2+2.
The natural response is: we multiply by x, so that is what we put above the line.

x The numbers we multiply the factor by to return the numerator terms
$x-1\overline{)x^2+0x+2}$
x^2-x The result when we multiply $x(x-1)$

We then subtract (x^2-x) from (x^2+0x+2)

$$\begin{array}{rr} & x^2+0x+2 \\ - & (x^2-x+0) \\ \hline & x+2 \end{array}$$

Then, we apply the same analysis to our result:

$x-1\overline{)x+2}$
We multiply $x-1$ by 1

$$\begin{array}{r} 1 \\ x-1\overline{)x+2} \\ x-1 \end{array}$$

Subtract $x-1$ from $x+2$

$$\begin{array}{rr} & x+2 \\ - & (x-1) \\ \hline & 3 \end{array}$$

This yields a remainder of 3, and our rational expression $\frac{x^2+2}{x-1} = x+1+\frac{3}{x-1}$. Each piece of this can be integrated directly.

$$\int \left(x+1+\frac{3}{x-1}\right)dx = \frac{x^2}{2}+x+3\ln|x-1|+C$$ ■

Let's do another example.

■ **Example 5.6** Evaluate $\int \frac{x^3+1}{x+1}dx$

First, we apply the long division process to simplify the expression algebraically so it can be integrated.

$$x+1\overline{)x^3+1}$$

In order to get x^3, we multiply x by x^2

$$\begin{array}{r} x^2 \\ x+1\overline{)x^3+1} \\ x^3+x^2 \end{array}$$

Subtract x^3+x^2 from x^3+1

$$\begin{array}{rr} & x^3+0x^2+0x+1 \\ - & (x^3+x^2+0x+0) \\ \hline & -x^2+1 \end{array}$$

Repeat the process with our result

$$x+1\overline{)-x^2+1}$$

We multiply x by a $-x$ to get $-x^2$

$$\begin{array}{r} -x \\ x+1\overline{)-x^2+1} \\ -x^2-x \end{array}$$

Subtract $-x^2-x$ from $-x^2+1$

$$\begin{array}{rr} & -x^2+0x+1 \\ - & (-x^2-x+0) \\ \hline & x+1 \end{array}$$

We can, again, repeat the process with our result

$x+1\overline{)x+1}$, but since this is the same as $\dfrac{x+1}{x+1}$, we can see that it reduces to 1.

There is no remainder, so the term $x+1$ is a factor of x^3+1, and the rational expression $\dfrac{x^3+1}{x+1} = x^2-x+1$.

We integrate this to define $\int \dfrac{x^3+1}{x+1}dx = \int \left(x^2-x+1\right)dx = \dfrac{x^3}{3} - \dfrac{x^2}{2} + x + C$

If the original integral were given bounds, $\int_0^1 \dfrac{x^3+1}{x+1}dx = \dfrac{x^3}{3} - \dfrac{x^2}{2} + x|_0^1 = \dfrac{1}{3} - \dfrac{0^3}{3} - \left(\dfrac{1^2}{2} - \dfrac{0^2}{2}\right) + 1 - 0 = \dfrac{5}{6}$ ■

5.1.4 Completing the Square

Recall: Transforming a polynomial ax^2+bx+c to a "square" $a(x-h)^2+k$ by completing the square. There are multiple approaches to result in the "square", first we will derive it.

■ **Example 5.7** Rewrite $f(x) = 2x^2+12x-3$ by completing the square.

First, we factor the a out of the x terms
$f(x) = 2(x^2+6x)-3$
Then, we determine a square of the form x^2+6x+C. To do this, we notice that $(x+m)^2 = x^2+2mx+m^2$, so if $2mx = 6x$, then $m = 3$. So, $m^2 = 9$.
We want to find the equivalent to x^2+6x, so we subtract the m^2 term from $(x+3)^2$.
$f(x) = 2\left((x+3)^2-9\right)-3$
Simplify, $f(x) = 2(x+3)^2-18-3 = 2(x+3)^2-21$. ■

For a general form, $a(x-h)^2+k$, a is the same a in ax^2+bx+c. $h = \dfrac{-b}{2a}$, and $k = \dfrac{-b^2}{4a}+c$.

Check: $h = \dfrac{-12}{2*2} = -3$ ✓, $k = \dfrac{-144}{4*2}-3 = -18-3 = -21$ ✓.
This also yields $f(x) = 2(x+3)^2-21$.

We apply this technique to simplify expressions in integrals so we are able to integrate them.

■ **Example 5.8** Evaluate $\int \dfrac{x+2}{x^2+4x+8}dx$
Here, the denominator is of a higher order than the numerator, so we cannot apply long division to simplify. We seek to complete the square in the denominator so we may be able to integrate it.
Complete the square for: $f(x) = x^2+4x+8$.
Whether you work through the process step-by-step, or use the formulas, we arrive at $f(x) = (x+2)^2+4$.
$\int \dfrac{x+2}{(x+2)^2+4}dx$ we can simplify first by a u-substitution $u = x+2$, $du = dx$.
$\int \dfrac{u}{u^2+4}du$ we can simplify by a second u-substitution $w = u^2+4$, $dw = 2udu$.
$\dfrac{1}{2}\int \dfrac{dw}{w} = \dfrac{1}{2}\ln|w|+C = \dfrac{1}{2}\ln|u^2+4|+C = \dfrac{1}{2}\ln|(x+2)^2+4|+C$ ■

■ **Example 5.9** Evaluate $\int \dfrac{x+1}{x^2-2x+5}dx$
We simplify the integral by completing the square on the denominator: $f(x) = x^2-2x+5 = (x-1)^2+4$.
$\int \dfrac{x+1}{(x-1)^2+4}dx$, we can evaluate using a u-substitution $u = x-1$, $du = dx$. This implies that $x = u+1$, so $x+1 = u+2$.
$\int \dfrac{u+2}{u^2+4}du$, which we can simplify by separating the numerator terms.
$\int \dfrac{u}{u^2+4}du + \int \dfrac{2}{u^2+4}du$
The first term we can integrate using a second u-substitution, $w = u^2+4$, $dw = 2udu$.
$\dfrac{1}{2}\int \dfrac{dw}{w} = \dfrac{1}{2}\ln|w|+C = \dfrac{1}{2}\ln|u^2+4|+C = \dfrac{1}{2}\ln|(x-1)^2+4|+C$.
The second term is a recognizable form from your known derivatives, recall: $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{x^2+1}$
Factor 4 out of the denominator: $u^2+4 = 4\left(\dfrac{u^2}{4}+1\right)$
$\int \dfrac{2}{4(u^2/4+1)}du = \dfrac{1}{2}\int \dfrac{1}{u^2/4+1}du$
We can evaluate this with another u-substitution $t = u/2$, $dt = \dfrac{1}{2}du$
$\int \dfrac{1}{t^2+1}dt = \arctan(t)+C = \arctan\left(\dfrac{u}{2}\right)+C = \arctan\left(\dfrac{x-1}{2}\right)+C$

Combining the results yields $\int \frac{x+1}{x^2-2x+5}dx = \frac{1}{2}\ln|(x-1)^2+4| + \arctan\left(\frac{x-1}{2}\right) + C$ ■

5.2 Integration by Parts

Integration by Parts is an integration technique that effectively "undoes" the Product Rule for derivatives. It is an appropriate technique for most problems involving the multiplication of two functions.

Recall: The Product Rule
$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

We know by the Fundamental Theorem of Calculus that $\int \frac{d}{dx}(f(x)g(x))\,dx = f(x)g(x)$, so we know that $\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)g(x)$. This is what we derive the Integration by Parts technique from.

$\int f'(x)g(x)dx + \int f(x)g'(x)dx = f(x)g(x)$, we move the first term to the right side of the equation to define the form of this technique:
$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

We seek functions of the form on the left side of this equation in order to apply the technique.

A general form that is generally used for memorization, etc. uses $u(x)$ and $v(x)$ in place of $f(x)$ and $g(x)$.

$\int udv = uv - \int vdu$

5.2.1 Choosing u and v

Generally, you want u to "go away" as you take more derivatives (this is certainly NOT always the case). Polynomials are a good choice for u (though again, NOT always).

If one of your functions is hard to integrate, or otherwise really messy, it should probably be chosen as your u. Logarithms are nearly always u, because we currently do not have an antiderivative for logarithms. (In fact, we'll use Integration by Parts to define the antiderivative of logarithms).

Important notes:

1. DO NOT switch u and v ever. Doing so will only reverse the last step of integration by parts.
2. Be very careful about your signs $\pm$
3. It is simple to check your answer: just differentiate it! (This is true for all integration techniques)

■ **Example 5.10** Evaluate $\int x^2\cos(x)dx$
This is a product of two functions, so we need to select which function will be "u" and which will become the "dv". We select $u = x^2$ because derivatives of it will eventually reduce to a constant (the third derivative specifically). Thus $dv = \cos(x)dx$ (this form is important for maintaining consistency!).
We apply the form for Integration by Parts: $\int udv = uv - \int vdu$
Since $u = x^2$, $du = 2xdx$. Since $dv = \cos(x)dx$, $v = \int \cos(x)dx = \sin(x)$ (we do not use the $+C$

until the very end of the problem).
Thus, $\int x^2\cos(x)dx = x^2\sin(x) - \int \sin(x)*2xdx = x^2\sin(x) - \int 2x\sin(x)dx$
We then apply integration by parts a second time to the integral $\int 2x\sin(x)dx$.
We use $u = 2x$, and $dv = \sin(x)dx$ to maintain consistency in our method. Thus, $du = 2dx$, and $v = -\cos(x)$.
$\int 2x\sin(x)dx = -2x\cos(x) - \int -\cos(x)*2dx = -2x\cos(x) + \int 2\cos(x)dx$
We can integrate this last term directly.
$\int 2x\sin(x)dx = -2x\cos(x) + 2\sin(x)$.
And, $\int x^2\cos(x)dx = x^2\sin(x) - (-2x\cos(x) + 2\sin(x)) + C = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$
Check: $\frac{d}{dx}\left(x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C\right) = 2x\sin(x) + x^2\cos(x) + 2\cos(x) - 2x\sin(x) - 2\cos(x) = x^2\cos(x)$✓

■

Trickier example:

■ **Example 5.11** Evaluate $\int \cos(x)\sin(x)dx$ using Integration by Parts
Since these are both trigonometric functions, we can select u and dv arbitrarily.
$u = \sin(x)$, $dv = \cos(x)dx$, so $du = \cos(x)dx$ and $v = \sin(x)$
Since neither of these functions will simplify to a constant, we will want to keep track of both sides through the entire analysis.
$\int \cos(x)\sin(x)dx = \sin(x)\sin(x) - \int \sin(x)\cos(x)dx$

Note that $\int \cos(x)\sin(x)dx = \int \sin(x)\cos(x)dx$, so if we add it to both sides we have:
$2\int \cos(x)\sin(x)dx = \sin^2(x)$
And $\int \cos(x)\sin(x)dx = \frac{1}{2}\sin^2(x) + C$.(If you flip the choices of u and dv, then you arrive at $\frac{-1}{2}\cos^2(x) + C$. These are equivalent due to the Pythagorean identity: $\cos^2(x) + \sin^2(x) = 1$.)

Check: $\frac{d}{dx}\left(\frac{1}{2}\sin^2(x) + C\right) = \frac{2}{2}\sin(x)\cos(x) = \cos(x)\sin(x)$ ✓ ■

Keep track of your choices for u and dv so that you continue to make progress with each step.

5.2.2 Definite Integrals with Integration by Parts

■ **Example 5.12** Evaluate $\int_0^{\ln(2)} xe^x dx$
We select $u = x$, and $dv = e^x dx$. So, $du = dx$, and $v = e^x$.
Applying the form of Integration by Parts:
$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x$
Evaluate using the Fundamental Theorem of Calculus:
$\int_0^{\ln(2)} xe^x dx = xe^x - e^x|_0^{\ln(2)} = \ln(2)e^{\ln(2)} - e^{\ln(2)} - \left(0e^0 - e^0\right) = 2\ln(2) - 2 + 1 = 2\ln(2) - 1$ ■

■ **Example 5.13** Evaluate $\int_1^4 \ln(x)dx$
We do not have a defined antiderivative for $\ln(x)$, so we select $u = \ln(x)$ and $dv = dx$; $du = \frac{1}{x}dx$, and $v = x$.
Applying the form of Integration by Parts:
$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - \int dx = x\ln(x) - x$
Evaluate using the Fundamental Theorem of Calculus:
$\int_1^4 \ln(x)dx = x\ln(x) - x|_1^4 = 4\ln(4) - 4 - (1*0 - 1) = 4\ln(4) - 4 + 1 = 4\ln(4) - 3$. ■

5.3 Trigonometric Integrals

Trigonometric integrals are any integrals involving trigonometric functions. This section is particularly focused on using the trigonometric identities to simplify trigonometric integrals in order to evaluate them.

Trigonometric identities you need to know:
$\sin^2(x)+\cos^2(x)=1$
$\sin^2(x)=\frac{1}{2}(1-\cos(2x))$
$\cos^2(x)=\frac{1}{2}(1+\cos(2x))$
$\sin(2x)=2\sin(x)\cos(x)$
$\cos(2x)=\cos^2(x)-\sin^2(x)$
$1+\tan^2(x)=\sec^2(x)$
$\sin(a+b)=\sin(a)\cos(b)+\sin(b)\cos(a)$
$\sin(a-b)=\sin(a)\cos(b)-\sin(b)\cos(a)$
$\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$
$\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$

Important derivatives of trigonometric functions:
$\frac{d}{dx}(\sin(x))=\cos(x)$
$\frac{d}{dx}(\cos(x))=-\sin(x)$
$\frac{d}{dx}(\tan(x))=\sec^2(x)$
$\frac{d}{dx}(\sec(x))=\sec(x)\tan(x)$

It is also important to recall your product and quotient derivative rules, in order to appropriately determine derivatives of other trigonometric functions.

With appropriate substitutions and manipulation, you can evaluate an integral of almost any combination of trigonometric functions.

The nicest possible scenario for multiple trigonometric functions is one where you can use a u-substitution. To help you strategize, these are most commonly the forms:

$\int \sin^m(x)\cos^n(x)dx$, where n is odd, and m is even. In these cases, we can use $\sin^2(x)+\cos^2(x)=1$ and a u-substitution of $u=\sin(x)$, $du=\cos(x)dx$.

Example 5.14 Evaluate $\int \sin^4(x)\cos^3(x)dx$
We note that 4 is even, and 3 is odd, so the u-substitution is $u=\sin(x)$, $du=\cos(x)dx$
This allows us to treat $\sin^4(x)=u^4$, and $\cos(x)dx=du$, but we still have a $\cos^2(x)$ term that we need to rewrite in terms of u. By applying our pythagorean identity, $\sin^2(x)+\cos^2(x)=1$, we see that $\cos^2(x)=1-\sin^2(x)=1-u^2$.

So, the simplified integral becomes $\int u^4(1-u^2)du=\int(u^4-u^6)du=\frac{u^5}{5}-\frac{u^7}{7}+C$
When we substitute our functions of x back into this result, we have our antiderivative: $\frac{\sin^5(x)}{5}-\frac{\sin^7(x)}{7}+C$ ■

$\int \cos^m(x)\sin^n(x)dx$, where n is odd, and m is even. In these cases, we can use $\sin^2(x)+\cos^2(x)=1$ and a u-substitution of $u=\cos(x)$, $du=-\sin(x)dx$.

Example 5.15 Evaluate $\int \cos^2(x)\sin^5(x)dx$
We note that 2 is even, and 5 is odd, so the u-substitution is $u=\cos(x)$, $du=-\sin(x)dx$
This allows us to write $\cos^2(x)=u^2$, and $\sin(x)dx=-du$, but we still have a $\sin^4(x)$ term that

we need to rewrite in terms of u. By applying our pythagorean identity, $\sin^2(x)+\cos^2(x)=1$, we see that $\sin^4(x)=\left(\sin^2(x)\right)^2=\left(1-\cos^2(x)\right)^2=\left(1-u^2\right)^2$.
So, the simplified integral becomes $\int u^2\left(1-u^2\right)^2(-)du=-\int u^2\left(1-2u^2+u^4\right)du=-\int\left(u^2-2u^4+u^6\right)du=-\left(\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right)+C$
When we distribute the $-$, and substitute our functions of x back into this result, we have the antiderivative: $\frac{-\cos^3(x)}{3}+\frac{2\cos^5(x)}{5}-\frac{\cos^7(x)}{7}+C$ ■

$\int\tan^n(x)\sec^m(x)dx$, where n is any integer, and m is even. In these cases, we can use $1+\tan^2(x)=\sec^2(x)$ and a u-substitution of $u=\tan(x)$, $du=\sec^2(x)dx$.

■ **Example 5.16** Evaluate $\int\tan^2(x)\sec^4(x)dx$
We note that the power of $\sec(x)$ is even, so we apply a u-substitution of $u=\tan(x)$, $du=\sec^2(x)dx$
This allows us to write $\tan^2(x)=u^2$, and $\sec^2(x)dx=du$, but we still have a $\sec^2(x)$ term that we need to rewrite in terms of u. By applying the identity $1+\tan^2(x)=\sec^2(x)$, we see that $\sec^2(x)=1+\tan^2(x)=1+u^2$.
So the simplified integral becomes $\int u^2(1+u^2)du=\int(u^2+u^4)du=\frac{u^3}{3}+\frac{u^5}{5}+C$
When we substitute our functions of x back into this result, we have the antiderivative: $\frac{\tan^3(x)}{3}+\frac{\tan^5(x)}{5}+C$ ■

$\int\tan^n(x)\sec^m(x)dx$, where n is odd, and m is any integer. In these cases, we can use $1+\tan^2(x)=\sec^2(x)$ and a u-substitution of $u=\sec(x)$, $du=\sec(x)\tan(x)dx$. Note: there is significant overlap between this case and the last case - if that happens, use the one you are most comfortable with.

■ **Example 5.17** Evaluate $\int\tan^5(x)\sec^3(x)dx$
We note that the power of $\tan(x)$ is odd, so we apply a u-substitution of $u=\sec(x)$, $du=\sec(x)\tan(x)dx$
This allows us to write $\sec^2(x)=u^2$ (this is because one term of $\sec(x)$ must be factored out for du), and $\tan(x)\sec(x)dx=du$. However, we still have the term $\tan^4(x)$ that we need to rewrite in terms of u. We apply the identity $1+\tan^2(x)=\sec^2(x)$ to see that $\tan^4(x)=\left(\tan^2(x)\right)^2=\left(\sec^2(x)-1\right)^2=\left(u^2-1\right)^2$.
The simplified integral becomes $\int\left(u^2-1\right)^2u^2du=\int\left(u^4-2u^2+1\right)u^2du=\int\left(u^6-2u^4+u^2\right)du=\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}+C$
When we substitute our functions of x back into this result, we have the antiderivative: $\frac{\sec^7(x)}{7}-\frac{2\sec^5(x)}{5}+\frac{\sec^3(x)}{3}+C$ ■

You will have to practice several problems, and use the identities listed, in order to confidently solve more general problems.

■ **Example 5.18** Evaluate $\int\sin^2(x)\cos^4(x)dx$

This is not one of the cases above, both powers are even. So, we will have to use a few identities to reach the point where we can integrate it exactly.

This is where confidence in your identities is exceptionally helpful. In this case, let's use the half-angle identities, $\sin^2(x)=\frac{1}{2}(1-\cos(2x))$ and $\cos^2(x)=\frac{1}{2}(1+\cos(2x))$. Note: This is not the only way to solve this, try another and see if your answer is consistent!

$$\int \frac{1}{2}(1-\cos(2x))\left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$$

Then simplify/expand

$\int \frac{1}{8}\left(1-\cos^2(2x)\right)(1+\cos(2x))\,dx$

$\int \frac{1}{8}\left(\sin^2(2x)\right)(1+\cos(2x))\,dx$

$\int \frac{1}{16}(1-\cos(4x))(1+\cos(2x))\,dx$

$\int \frac{1}{16}(1+\cos(2x)-\cos(4x)-\cos(4x)\cos(2x))\,dx$

The first three terms we can integrate as they are, but we need to use another set of identities to simplify $\cos(4x)\cos(2x)$, namely the sum and difference identities for cosine.

Since $\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$ and $\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$, $\cos(a+b)+\cos(a-b)=2\cos(a)\cos(b)$, we can define $\cos(4x)\cos(2x)=\frac{1}{2}(\cos(6x)+\cos(2x))$

$\int \frac{1}{16}\left(1+\cos(2x)-\cos(4x)-\frac{1}{2}(\cos(6x)+\cos(2x))\right)dx$

$=\frac{1}{16}\left(x+\frac{1}{2}\sin(2x)-\frac{1}{4}\sin(4x)-\frac{1}{12}\sin(6x)-\frac{1}{4}\sin(2x)\right)+C$ ■

Adding additional terms: u-substitution and a trigonometric integral, integration by parts and a trigonometric integral, etc.

■ **Example 5.19** Evaluate $\int x\sec^2(x^2)\tan^4(x^2)dx$

We first notice that the trigonometric functions are functions of x^2 instead of x, so we need to apply a u-substitution first to simplify them to known forms of the trigonometric functions.

$u=x^2$, $du=2xdx$.

This simplifies our integral to only a trigonometric integral $\int \frac{1}{2}\sec^2(u)\tan^4(u)du$

We notice that both functions have even powers, so we will use the third case where $w=\tan(u)$, and $dw=\sec^2(u)du$, there are no additional terms to convert to w's.

$\int \frac{1}{2}w^4dw=\frac{1}{2}\frac{w^5}{5}+C=\frac{1}{10}\tan^5(u)+C=\frac{1}{10}\tan(x^2)+C$ ■

This will take practice through trial and error, and perserverance. Confidence in your trigonometric identities and algebra skills will get you through this. Practice!

5.4 Trigonometric Substitution

Trigonometric substitution uses a substitution of $x=f(\theta)$ where f is a trigonometric function. This is a substitution in the opposite direction of a u-substitution. Instead of seeking a function that is in the integral with its derivative, we replace all x's with $f(\theta)$'s, and dx with $f'(\theta)d\theta$. We use this technique for three cases:

1. $\sqrt{a^2-x^2}$: In this case, we apply a substitution $x=a\sin(\theta)$. If we relate this to our knowledge of trigonometric functions and triangles from trigonometry, this relation is equivalent to $\frac{x}{a}=\sin(\theta)=\frac{\text{opposite}}{\text{hypotenuse}}$. This substitution is helpful because when we replace x with $a\sin(\theta)$, $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2(\theta)}=a\sqrt{1-\sin^2(\theta)}=a\cos(\theta)$. This is a function we can integrate, which is why the technique is a helpful approach.
2. $\sqrt{a^2+x^2}$: In this case, we apply a substitution $x=a\tan(\theta)$. If we relate this to our knowledge of trigonometric functions and triangles from trigonometry, this relation is equivalent to

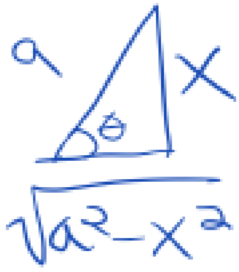


Figure 5.1: The right triangle for the $a\sin(\theta)$ substitution. The hypotenuse is a, the opposite side from θ is x, and using the Pythagorean Theorem we determine the adjacent side is $\sqrt{a^2-x^2}$.

$\frac{x}{a} = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$. This substitution is helpful because when we replace x with $a\tan(\theta)$,

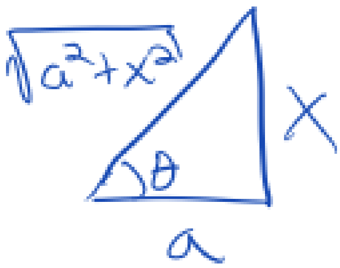


Figure 5.2: The right triangle for the $a\tan(\theta)$ substitution. The opposite side from θ is x, the adjacent side is a, and using the Pythagorean Theorem we determine the hypotenuse is $\sqrt{a^2+x^2}$.

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2(\theta)} = a\sqrt{1+\tan^2(\theta)} = a\sec(\theta)$.

3. $\sqrt{x^2-a^2}$: In this case, we apply a substitution $x = a\sec(\theta)$. If we relate this to our knowledge of trigonometric functions and triangles from trigonometry, this relation is equivalent to $\frac{x}{a} = \sec(\theta)$, or $\frac{a}{x} = \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$. This substitution is helpful because when we

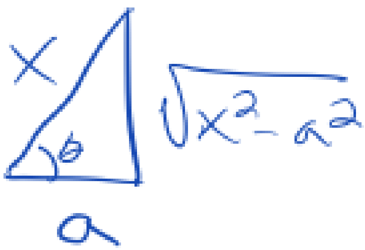


Figure 5.3: The right triangle for the $a\sec(\theta)$ substitution. The hypotenuse is x, the adjacent side to θ is a, and using the Pythagorean Theorem we determine the opposite side is $\sqrt{x^2-a^2}$.

replace x with $a\sec(\theta)$, $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2(\theta)-a^2} = a\sqrt{\sec^2(\theta)-1} = a\tan(\theta)$.

The key to these problems is confidence in trigonometric identities and triangles. The forms of the substitutions directly come from the identities.

■ **Example 5.20** Evaluate $\int \frac{1}{x^2\sqrt{1+x^2}}dx$

First, we recognize the form of the square root, $\sqrt{1+x^2}$ is of the form $\sqrt{a^2+x^2}$ where $a = 1$. So,

the substitution is $x=\tan(\theta)$, and $dx=\sec^2(\theta)d\theta$.
Replacing each x with $\tan(\theta)$, and dx with $\sec^2(\theta)d\theta$.
$\int \frac{1}{\tan^2(\theta)\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)d\theta = \int \frac{\sec^2(\theta)}{\tan^2(\theta)\sec(\theta)}d\theta$
$= \int \frac{\sec(\theta)}{\tan^2(\theta)}d\theta = \int \frac{\frac{1}{\cos(\theta)}}{\frac{\sin^2(\theta)}{\cos^2(\theta)}}d\theta = \int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta$
Now, it is in a form where we can apply a u-substitution, $u=\sin(\theta)$, $du=\cos(\theta)d\theta$
$\int \frac{1}{u^2}du = \frac{-1}{u}+C = \frac{-1}{\sin(\theta)}+C$
However, we need to convert from θ back to x. We know $x=\tan(\theta)$, and we can use that to construct the triangle, and then use the triangle to define $\sin(\theta)=\frac{\text{opposite}}{\text{hypotenuse}}=\frac{x}{\sqrt{1+x^2}}$, so $\frac{-1}{\sin(\theta)}=\frac{-\sqrt{1+x^2}}{x}$.
$\int \frac{1}{x^2\sqrt{1+x^2}}dx = \frac{-\sqrt{1+x^2}}{x}+C$ ■

■ **Example 5.21** Evaluate $\int \frac{dx}{x^2\sqrt{x^2-4}}$
First, we recognize the form of the square root in the integral, $\sqrt{x^2-4}$ is of the form $\sqrt{x^2-a^2}$, where $a=2$. So, the substitution is $x=2\sec(\theta)$, and $dx=2\sec(\theta)\tan(\theta)d\theta$.
$\int \frac{2\sec(\theta)\tan(\theta)d\theta}{4\sec^2(\theta)\sqrt{4\sec^2(\theta)-4}} = \int \frac{1}{4}\cos(\theta)d\theta = \frac{1}{4}\sin(\theta)+C$
Again, we need to convert back to x terms, and we use the triangle to determine that $\sin(\theta)=\frac{\sqrt{x^2-4}}{x}$.
$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{x}+C$ ■

■ **Example 5.22** Evaluate $\int \sqrt{1-4x^2}dx$
First, we recognize the form of the square root in the integral, $\sqrt{1-4x^2}=2\sqrt{\frac{1}{4}-x^2}$ is of the form $\sqrt{a^2-x^2}$ with $a=\frac{1}{2}$. So, the substitution is $x=\frac{1}{2}\sin(\theta)$, and $dx=\frac{1}{2}\cos(\theta)d\theta$.
$\int \sqrt{1-4\frac{1}{4}\sin^2(\theta)}\frac{1}{2}\cos(\theta)d\theta = \int \frac{1}{2}\cos^2(\theta)d\theta$
Now, we have a trigonometric integral, and we need to use another identity to integrate it: $\cos(\theta)=\frac{1}{2}(1+\cos(2\theta))$ $\int \frac{1}{2}\cos^2(\theta)d\theta = \int \frac{1}{4}(1+\cos(2\theta))d\theta = \frac{1}{4}\theta+\frac{1}{8}\sin(2\theta)+C$
Again, we need to convert back to x terms, so we need to use another identity $\sin(2\theta)=2\sin(\theta)\cos(\theta)$, and solve for θ in our initial substitution.
$\theta=\arcsin(2x)$, and $2\sin(\theta)\cos(\theta)=2(2x)\sqrt{1-4x^2}$
So, $\int \sqrt{1-4x^2}dx = \frac{1}{4}\arcsin(2x)+\frac{1}{2}x\sqrt{1-4x^2}+C$ ■

As you notice from the last example, they will not always look exactly as the forms suggest. However, through some algebraic manipulation, we can write them in the appropriate form to evaluate them using the standard trigonometric substitution. Practice these algebraic manipulations, and check your answers with the results given.

$\sqrt{9x^2+4} \to 3\sqrt{x^2+\frac{4}{9}}$

$\sqrt{16x^2-25} \to 4\sqrt{x^2-\frac{25}{16}}$

$\sqrt{49-\frac{x^2}{4}} \to \frac{1}{2}\sqrt{196-x^2}$

Completing the Square

Recall: Completing the Square

We can use completing the square to convert the square root of a quadratic function into a form that we can solve using a trigonometric substitution.

■ **Example 5.23** Evaluate $\int \frac{1}{\sqrt{x^2+4x-12}}dx$

First, we realize that we cannot apply a u-substitution: no derivative present

We should not apply integration by parts: This is a messy rational function, so even if we make it our‘u’, the result may not be integrable.

Since we have a quadratic inside of a square root, that hints at a trigonometric substitution. However, it’s not in the right form. So, we complete the square to get it in the right form.

$x^2+4x-12$, I notice that the middle term is $+4$, and when we expand a square the integer is doubled in our x term ($(x+h)^2 = x^2+2hx+h^2$). So, we know we have something of the form $(x+2)^2+k$. Solve for that k: $x^2+4x-12 = x^2+4x+4+k$, and so $k=-16$.

The form we replace $x^2+4x-12$ with, is $(x+2)^2-16$.

Now we evaluate $\int \frac{1}{\sqrt{(x+2)^2-16}}dx$

This is in the form of a Secant trigonometric substitution with $a=4$, but it has that $x+2$ piece... do a u-substitution first. $u=x+2$, $du=dx$.

$\int \frac{1}{\sqrt{u^2-16}}du$

Now, we apply our trigonometric substitution $x=4\sec(\theta)$, $dx=4\sec(\theta)\tan(\theta)d\theta$.

$\int \frac{1}{\sqrt{16\sec^2(\theta)-16}}4\sec(\theta)\tan(\theta)d\theta$

Simplify: $\int \frac{4\sec(\theta)\tan(\theta)}{4\tan(\theta)}d\theta = \int \sec(\theta)d\theta$

We can integrate this using a multiply-by-one: $\sec(\theta)+\tan(\theta)$

$\int \frac{\sec(\theta)+\tan(\theta)}{\sec(\theta+\tan(\theta)}\sec(\theta)d\theta = \int \frac{\sec^2(\theta)+\sec(\theta)\tan(\theta)}{\sec(\theta)+\tan(\theta)}d\theta$

Now we can recognize that the numerator is the derivative of the denominator, so we can apply a u-substitution, $w=\sec(\theta)+\tan(\theta)$, and $dw = \left(\sec(\theta)\tan(\theta)+\sec^2(\theta)\right)d\theta$.

$\int \frac{dw}{w} = \ln|w|+C = \ln|\sec(\theta)+\tan(\theta)|+C$

Converting to u and then x:

$\ln|\frac{u}{4}+\frac{\sqrt{u^2-16}}{4}|+C = \ln|\frac{x+2}{4}+\frac{\sqrt{(x+2)^2-16}}{4}|+C$ ■

5.5 Partial Fraction Decomposition

Partial fraction decomposition is a technique used to integrate rational expressions involving quadratics (or higher) that are factorable. The concept behind this technique is that we already know how to integrate expressions like $\frac{1}{u}$, and if you recall from algebra: when we add fractions, the denominators are multiplied. For two rational expressions, this might look like $\frac{1}{u}+\frac{1}{v}=\frac{v+u}{uv}$.

This form arrives at a factorable quadratic function when the denominators are linear functions of x.

■ **Example 5.24** Adding the fractions $\frac{1}{x+3}$ and $\frac{3}{2x-1}$, using algebra:
$\frac{1}{x+3}+\frac{3}{2x-1}=\frac{(2x-1)+3(x+3)}{(x+3)(2x-1)}=\frac{5x+8}{(x+3)(2x-1)}$ ■

This is the process we want to invert through partial fraction decomposition.

The Process: 1) Factor the denominator (if it is not factorable, complete the square and apply a trigonometric substitution)
2) Use the factors, $(x+C)$ and $(x+D)$ to set up the form for partial fraction decomposition:
$\frac{A}{x+C}+\frac{B}{x+D}=\frac{A(x+C)+B(x+D)}{(x+C)(x+D)}$ (A and B are undefined)
3) Solve algebraically for the coefficients A and B to rewrite the rational expression in an integrable form

■ **Example 5.25** Use partial fraction decomposition to rewrite $\frac{3}{2x^2+3x+1}$ in an integrable form
1) Factor $2x^2+3x+1$: $(2x+1)(x+1)$
2) Set up the form: $\frac{A}{2x+1}+\frac{B}{x+1}=\frac{A(x+1)+B(2x+1)}{(2x+1)(x+1)}$
3) Solve algebraically for A and B: $\frac{Ax+2Bx+A+B}{(2x+1)(x+1)}=\frac{3}{(2x+1)(x+1)}$
Note that the denominators are already the same, so we only need to match the numerators:
$Ax+2Bx+A+B=3$
Notice that there are no terms of x on the right side, so $Ax+2Bx=0x$, or $A+2B=0$
A and B are constants, so $A+B=3$
Now, we have a system of linear equations to solve for A and B
The first equation tells us that $A=-2B$, and we can substitute this in for B into the second equation.
$-2B+B=3\to -B=3\to B=-3$
Thus, $A=-2(-3)=6$, and $\frac{A}{2x+1}+\frac{B}{x+1}=\frac{6}{2x+1}-\frac{3}{x+1}$ (this is your integrable form - you will stop here and then integrate the functions you found)
We can <u>check</u> our work by adding the fractions: $\frac{6(x+1)-3(2x+1)}{(2x+1)(x+1)}=\frac{6x+6-6x-3}{(2x+1)(x+1)}=\frac{3}{(2x+1)(x+1)}$ ✓
■

Common places for errors:
Algebra in solving the linear systems - review this if it is not familiar!
Accidentally switching A and B - to avoid this, clearly write the form before you solve for the coefficients, and refer back to it when you write down your integrable form.

5.5.1 Applying Partial Fraction Decomposition to Evaluate an Integral

All of the new work is involved in the process of partial fraction decomposition. Once you are comfortable with that step, start practicing the full process to evaluate integrals.

■ **Example 5.26** Evaluate $\int \frac{2x}{x^2-\frac{1}{2}x-\frac{1}{2}}dx$
1) Factor $x^2-\frac{1}{2}x-\frac{1}{2}\to (x-1)(x+\frac{1}{2})$

2) Set up the form: $\dfrac{A}{x-1}+\dfrac{B}{x+\frac{1}{2}}=\dfrac{A(x+\frac{1}{2})+B(x-1)}{(x-1)(x+\frac{1}{2})}$

3) Solve algebraically for A and B: $\dfrac{Ax+Bx+\frac{1}{2}A-B}{(x-1)(x+\frac{1}{2})}=\dfrac{2x}{(x-1)(x+\frac{1}{2})}$

$Ax+Bx+\frac{1}{2}A-B=2x$

This turns into the system $A+B=2$ and $\frac{1}{2}A-B=0$

The second equation simplifies to $B=\frac{1}{2}A$

Substituting this into the first equation yields $A+\frac{1}{2}A=2\rightarrow\frac{3}{2}A=2\rightarrow A=\frac{4}{3}$

So, $B=\frac{2}{3}$, and $\dfrac{4/3}{x-1}+\dfrac{2/3}{x+\frac{1}{2}}$

4) Integrate the result: $\int\left(\dfrac{4/3}{x-1}+\dfrac{2/3}{x+\frac{1}{2}}\right)dx=\frac{4}{3}\int\dfrac{1}{x-1}dx+\frac{2}{3}\int\dfrac{1}{x+\frac{1}{2}}dx=\frac{4}{3}\ln|x-1|+\frac{2}{3}\ln|x+\frac{1}{2}|+C$ ■

■ **Example 5.27** Evaluate $\int\dfrac{x-1}{2x^2+3x+1}dx$

1) Factor $2x^2+3x+1$: $(2x+1)(x+1)$

2) Set up the form: $\dfrac{A}{2x+1)}+\dfrac{B}{x+1}=\dfrac{A(x+1)+B(2x+1)}{(x+1)(2x+1)}$

3) Solve algebraically for A and B: $\dfrac{Ax+2Bx+A+B}{(x+1)(2x+1)}=\dfrac{x-1}{(x+1)(2x+1)}$

$Ax+2Bx+A+B=x-1$, so $Ax+2Bx=x\rightarrow A+2B=1$ and $A+B=-1$

$A=1-2B$, so $1-2B+B=-1\rightarrow 1-B=-1\rightarrow 2=B$

$B=2$ and $A+2=-1\rightarrow A=-3$

$\dfrac{-3}{2x+1}+\dfrac{2}{x+1}$

4) Integrate: $\int\left(\dfrac{-3}{2x+1}+\dfrac{2}{x+1}\right)dx=-3\int\dfrac{1}{2x+1}dx+2\int\dfrac{1}{x+1}dx=\dfrac{-3}{2}\ln|2x+1|+2\ln|x+1|+C$ ■

5.5.2 Repeated Linear Factors

When we have a repeated term in the denominator, as in $\dfrac{2}{x^3(x-1)^2}$, we will need to expand to more terms to incorporate the combinations of powers.

■ **Example 5.28** Expand $\dfrac{2}{x^3(x-1)^2}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}+\dfrac{E}{(x-1)^2}$

We need as many terms as the power in the denominator: 3 for a cubic, 2 for a square, etc.

Another change is in the algebra we use to solve for the coefficients: instead of adding all the fractions together, we multiply both sides by the denominator of the original expression.

$2=x^3(x-1)^2\left(\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}+\dfrac{E}{(x-1)^2}\right)$

Then, we expand and simplify to solve for the coefficients, as before.

$2 = Ax^2(x-1)^2 + Bx(x-1)^2 + C(x-1)^2 + Dx^3(x-1) + Ex^3$
$2 = A(x^4-2x^3+x^2) + B(x^3-2x^2+x) + C(x^2-2x+1) + D(x^4-x^3) + Ex^3$
$2 = (A+D)x^4 + (-2A+B-D+E)x^3 + (A-2B+C)x^2 + (B-2C)x + C$
To solve algebraically, match terms of x
x^4: $0x^4 = (A+D)x^4 \to 0 = A+D$
x^3: $0x^3 = (-2A+B-D+E)x^3 \to 0 = -2A+B-D+E$
x^2: $0x^2 = (A-2B+C)x^2 \to 0 = A-2B+C$
x: $0x = (B-2C)x \to 0 = B-2C$
1: $2 = C$
Now, we can back-substitute values to get each coefficient.
$C = 2$, so $0 = B-2(2) \to B = 4$
Substituting these into $0 = A-2B+C$ yields $0 = A-2(4)+2 \to 0 = A-6 \to A = 6$
Substituting A into $0 = A+D$ yields $0 = 6+D \to D = -6$
Then, substituting the known coefficients into $0 = -2A+B-D+E$ yields $0 = -2(6)+4-(-6)+E \to 0 = -2+E \to E = 2$
So, $A = 6$, $B = 4$, $C = 2$, $D = -6$, and $E = 2$.
$\frac{2}{x^3(x-1)^2} = \frac{6}{x} + \frac{4}{x^2} + \frac{2}{x^3} - \frac{6}{x-1} + \frac{2}{(x-1)^2}$ ■

Once we have the expanded form, we integrate normally: $\int \frac{2}{x^3(x-1)^2}dx = \int \left(\frac{6}{x} + \frac{4}{x^2} + \frac{2}{x^3} - \frac{6}{x-1} + \frac{2}{(x-1)^2}\right) dx$
$6\ln|x| - \frac{4}{x} - \frac{1}{x^2} - 6\ln|x-1| - \frac{2}{x-1} + C.$

5.5.3 Denominators Involving Irreducible Quadratic Functions

Irreducible quadratic functions are of the form x^2+a^2, so a rational expression involving an irreducible quadratic might look like $\frac{x^2-3x+3}{(x^2+2)(x-1)}$ (the denominator has a term of the form x^2+a^2). This is not a linear term, but we can still complete partial fraction decomposition to separate the irreducible quadratic from the linear term.

■ **Example 5.29** Apply partial fraction decomposition to $\frac{x^2-3x+3}{(x^2+2)(x-1)}$, and then integrate the result.
The denominator is already factored as much as possible, so we start by setting up the form. The form changes slightly with an irreducible quadratic: instead of a constant in the numerator, we use a linear term. The linear factor remains the same form as before.
$\frac{x^2-3x+3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$
Then, we solve for the coefficients algebraically, as before.
$x^2-3x+3 = (Ax+B)(x-1) + C(x^2+2) \to x^2-3x+3 = Ax^2 - Ax + Bx - B + Cx^2 + 2C \to x^2-3x+3 = (A+C)x^2 + (-A+B)x - B + 2C$
Matching terms, x^2: $1 = A+C$
x: $-3 = -A+B$
1: $3 = -B+2C$
We can use the elimination technique to add the last two equations together, yielding $-3+3 = -A+B-B+2C \to 0 = -A+2C \to A = 2C$.
Then, substitute this into the first equation, $1 = A+C \to 1 = 2C+C \to 1 = 3C \to \frac{1}{3} = C$.
Thus, $A = 2\frac{1}{3} = \frac{2}{3}$, and $-3 = -A+B \to -3 = \frac{-2}{3} + B \to \frac{-7}{3} = B$.

So, $A=\frac{2}{3}$, $B=\frac{-7}{3}$, and $C=\frac{1}{3}$.

$\frac{x^2-3x+3}{(x^2+2)(x-1)}=\frac{\frac{2}{3}x+\frac{-7}{3}}{x^2+2}+\frac{\frac{1}{3}}{x-1}$

Integrate: $\int\left(\frac{\frac{2}{3}x+\frac{-7}{3}}{x^2+2}+\frac{\frac{1}{3}}{x-1}\right)dx$

To evaluate the first term, we should separate the numerator: $\frac{\frac{2}{3}x+\frac{-7}{3}}{x^2+2}=\frac{2}{3}\frac{x}{x^2+2}-\frac{7}{3}\frac{1}{x^2+2}$.
The first of these terms we can use u-substitution to solve, and the second we can use a trigonometric substitution to solve.
Starting with the first term: $\int\frac{2}{3}\frac{x}{x^2+2}dx$, we apply a u-substitution $u=x^2+2$, $du=2xdx$.
$\int\frac{1}{3}\frac{1}{u}du=\frac{1}{3}\ln|u|+C=\frac{1}{3}\ln|x^2+2|+C$.
The second term: $\int\frac{-7}{3}\frac{1}{x^2+2}dx$, we apply a trigonometric substitution $x=\sqrt{2}\tan(\theta)$, $dx=\sqrt{2}\sec^2(\theta)d\theta$. $\int\frac{-7}{3}\frac{1}{2\tan^2(\theta)+2}\sqrt{2}\sec^2(\theta)d\theta=\int\frac{-7}{3}\frac{1}{\sqrt{2}}d\theta=\frac{-7}{3\sqrt{2}}\theta+C=\frac{-7}{3\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+C$.

The last term: $\int\frac{\frac{1}{3}}{x-1}dx=\frac{1}{3}\ln|x-1|+C$.
The additive constant is an arbitrary constant, so there is really only ONE $+C$ in the antiderivative.
We add the three terms together, yielding $\frac{1}{3}\ln|x^2+2|+\frac{-7}{3\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+\frac{1}{3}\ln|x-1|+C$. ■

5.6 Integration Strategies

Advice for integration techniques:
Most success in integration is built through practice, because it builds your pattern recognition to determine the appropriate integration technique to use. Ultimately, the key is to recognize the form of the integral you are given, and match it to the appropriate technique. If the integral does not appear to already be in a recognizeable form, see if you can manipulate it algebraically into a recognizable form (example: completing the square to complete a trigonometric substitution).
Check for a u-substitution first: it is one of the fastest techniques, but only works if you see a function and the form of its derivative.
Integration by parts: it is a commonly-needed technique in applications, and usually works if there is a product of two functions.
Trigonometric integrals: if there are no trigonometric functions, you know this isn't what you need.
Trigonometric substitution: there are only 3 forms, if it is one of them - go!, if it is none of them - try something else.
Partial fraction decomposition: if there is a rational function with a factorable denominator - this is your technique!
Keep in mind that <u>many</u> integrals will require <u>multiple techniques</u>.

5.7 Other Methods of Integration

For some integrals we cannot apply known techniques to evaluate them, and we have to turn to alternative methods.

5.7.1 Tables of Integrals

For these, you turn to the table of integrals. There is one in the back of nearly all calculus textbooks, and there are also several available online. Match the integral you are solving to the form in the table, and then apply the form for the antiderivative as shown in the table.

■ **Example 5.30** Evaluate $\int \frac{\sqrt{4-x^2}}{x^2}dx$
Note: The $4-x^2$ looks like a trigonometric substitution, so we CAN integrate it using a trigonometric substitution - however, we will use this as an example of applying the table. (A great exercise for you: apply the trigonometric substitution and check your answer.)
Searching the table, we find $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = \frac{-1}{x}\sqrt{a^2-x^2} - \arcsin\left(\frac{x}{a}\right) + C$.
So, we just need to match appropriately to define a. $a^2 = 4$, so $a = 2$, and $\int \frac{\sqrt{4-x^2}}{x^2}dx = \frac{-1}{x}\sqrt{4-x^2} - \arcsin\left(\frac{x}{2}\right) + C$ ■

■ **Example 5.31** Evaluate $\int \cos^4(x)dx$
Searching the table, we find $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$.
If we substitute in $n = 4$, we can then solve for the antiderivative.
$\int \cos^4(x)dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left(\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int dx\right) = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{8}\cos(x)\sin(x) + \frac{3}{8}x + C$ (we can also choose to use a trigonometric identity to integrate $\int \cos^2(x)dx$).
■

5.7.2 Computer Algebra Systems

Mathematica is an example of a computer algebra system (CAS), that computes integrals with an analytic system. We will use this for the Mathematica projects in class, and it is a resource for integrals that do not match the techniques available, nor exist in the table.

5.7.3 Numerical Methods

Numerical methods allow us to approximate the value of a definite integral. The values computed by most calculators and computers are actually computed numerically. We will discuss some techniques of numerical integration in the next section, so you will have more insight into how they work.

5.8 Numerial Integration

Numerical integration techniques allow us to approximate the value of an integral that is too complex for our available integration techniques. These techniques are built in a similar manner to our Riemann sums in Calculus I.

5.8.1 Error

Since our numerical techniques are approximations and not exact, they induce error (the difference between the approximation and the exact value). We can define the absolute error as the absolute value of the difference. If we call the approximation c, and the exact value x, this is $|x-c|$. The relative error is basically a percentage of the exact, and defined through $\frac{|x-c|}{|x|}$. Note that relative error blows up as $x \to 0$.

These computations allow us to evaluate how good our approximation is, in relation to the exact value.

5.8.2 Midpoint Rule

The midpoint rule is the numerical integration technique that is the most straight-forward conceptually, it is built in the same way as our midpoint Riemann sum. We use the midpoint x-value to define the height of the box, and build boxes to approximate the area under the curve by adding them together.

Method:
Use the number of points to define the gridpoint spacing, $\Delta x = \frac{b-a}{n}$
Define the grid (x-values on the interval) $x_k = a + k\Delta x$, for $k = 0, 1, 2, ..., n$
Define the midpoints: Each midpoint is defined through the average of the gridpoints $x^* = \frac{x_{k-1}+x_k}{2}$.
Evaluate the function in your integral, $\int f(x)dx$, at each of the midpoints $f(x^*)$.
Sum the values using the midpoint rule, $M(n) = \sum_{k=1}^{n} f(x^*)\Delta x$.

Doing these computations by hand is very cumbersome, so you will want to use software like Mathematica to help you solve the homework problems efficiently. Pay attention to the concept: How is it built? What does it tell you?

5.8.3 Trapezoidal Rule

The trapezoidal rule is built using the endpoints to define the area of a trapezoid under the curve. We then sum the trapezoids to approximate the area under the curve.

Recall: Area of a trapezoid $A = \frac{1}{2}(b_1 + b_2)h$, where in our problem the bases b_1 and b_2 are the heights at the gridpoints $f(x_{k-1})$ and $f(x_k)$, and the height h is the width of the trapezoid, Δx (it's on its side in relation to how you initially learn trapezoids). Therefore, the form for each trapezoid is $\frac{1}{2}(f(x_{k-1}) + f(x_k))\Delta x$.
When we add these all together, each value is repeated except the endpoints. Thus, if we add them all together, we have $T(n) = \left[\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right]\Delta x$

5.8.4 Simpson's Rule

Simpson's rule is built using three gridpoints, and approximating the curve through those three points with a parabola. This allows for curvature, whereas the previous two approximations used straight lines (midpoint is flat, and trapezoidal is a line between endpoints). The curvature, and the use of additional points increases the accuracy dramatically. The derivation of this method is developed further in Numerical Analysis, but for our class we will focus on the concept and use of the method. Due to the form of the method each "box" has a parabolic/curved top and uses three points, this means that the midpoint in that set of three gridpoints is only used for that "box". That midpoint is also the most heavily weighted of the three, and so it has the highest coefficient. Simpson's rule effectively uses two Δx boxes for each "box", and so this means we can only use it for an even number of gridpoints (even n).

The method, when all the pieces are added together is
$S(n) = [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3)...4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\frac{\Delta x}{3}$

We can track which term has which coefficient by breaking it down:
Endpoints have a coefficient of 1
Odd x_k (remember that k starts at 0) have a coefficient of 4
Even x_k have a coefficient of 2

5.8.5 Error of the Methods

Since all of these methods are approximations, we need to define and be able to quantify their errors. The error formulas are:
Midpoint $E_M \leq \frac{K(b-a)}{24}(\Delta x)^2$, where $K = \max f''(x)$ on $[a,b]$.
Trapezoidal $E_T \leq \frac{K(b-a)}{12}(\Delta x)^2$, where $K = \max f''(x)$ on $[a,b]$.
Simpson's $E_S \leq \frac{K(b-a)}{180}(\Delta x)^4$, where $K = \max f''''(x)$ on $[a,b]$.

Note: K in all of these cases is a constant. You can use the techniques from your max/min problems in Calculus I to determine the maximum on $[a,b]$.

5.8.6 Example Applying the Methods

■ **Example 5.32** Apply each method to approximate $\int_0^4 (x^4 - x^3)dx$ using $n = 4$

First, for all methods, we need to define the gridpoints. The interval $[a,b] = [0,4]$ (the bounds of the integral). Since $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4}{4} = 1$. Therefore, our gridpoints are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$ (Note: they will not usually be so nice...).

For our error comparison, we want to use the exact value $\int_0^4 \left(x^4 - x^3\right) dx = 140.8$

Midpoint: Define the midpoints: $\frac{x_0 + x_1}{2} = \frac{1}{2}$, $\frac{x_1 + x_2}{2} = \frac{3}{2}$, $\frac{x_2 + x_3}{2} = \frac{5}{2}$, and $\frac{x_3 + x_4}{2} = \frac{7}{2}$.

$M(4) = \left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right)\right]\Delta x = \left[\frac{1}{16} - \frac{1}{8} + \frac{81}{16} - \frac{27}{8} + \frac{625}{16} - \frac{125}{8} + \frac{2401}{16} - \frac{343}{8}\right] = \left[\frac{1554}{8} - \frac{496}{8}\right] = \frac{1058}{8} = 132.25$

Based on the error formula, we can bound the error (the error will be lower than the error bound if the method is implemented correctly).

First, we determine K. $f''(x) = 12x^2 - 6x$, and the interval is $[0,4]$. We want to find the maximum value of $f''(x)$ on this interval. Recall from Calculus I: a local maximum or minimum occurs when the derivative of the function is equal to zero. So, our first check is to take another derivative and set it equal to zero, if that value of x is in the interval, we evaluate the function value and compare it with the endpoints to determine the maximum.
$f'''(x) = 24x - 6 = 0$, $x = \frac{6}{24} = \frac{1}{4}$, and $\frac{1}{4}$ is in the interval $[0,4]$. So, we evaluate $f''(0) = 0$, $f''\left(\frac{1}{4}\right) = \frac{12}{16} - \frac{6}{4} = \frac{-3}{4}$, and $f''(4) = 192 - 24 = 168$. Clearly, the largest value of these three is 168, so $K = 168$.
We substitute this, a, b, and Δx into the error formula: $E_M \leq \frac{168(4-0)}{24}(1)^2 = 28$... this is a really large error bound.

However, the absolute error is $|140.8-132.25| = 8.55$, and the relative error is $\frac{8.55}{140.8} = 6.07\%$. So, this approximation is much better than its error bound implies.

If we did not know the exact value, the error bound implies that this technique is not going to perform well with only 4 gridpoints. We would improve the approximation by increasing n, or by using another technique.

Trapezoidal:
$T(4) = \left[\frac{1}{2}f(0)+f(1)+f(2)+f(3)+\frac{1}{2}f(4)\right]\Delta x = \frac{1}{2}(0-0)+1-1+16-8+81-27+\frac{1}{2}(256-64) = 158$

We determine the error bound with the same K as we used for Midpoint rule (they use the same derivative), $E_T \leq \frac{168(4-0)}{12}(1)^2 = 56$... this is worse than Midpoint rule (it is always double the error bound of Midpoint). This may seem counterintuitive, but using the endpoints is less accurate, generally, than using Midpoint.

The absolute error is $|140.8-158| = 17.2$, and the relative error is $\frac{17.2}{140.8} = 12.2\%$. Note that this is about double the error of Midpoint, as the error bound implies.

Again, if we did not know the exact value, the error bound implies that this technique is not going to perform well with only 4 gridpoints. We can improve this approximation by increasing n, or by using another technique: Simpson's.

Simpson's:
$S(4) = \frac{\Delta x}{3}[f(0)+4f(1)+2f(2)+4f(3)+f(4)] = \frac{1}{3}[0-0+4(1-1)+2(16-8)+4(81-27)+256-64] = \frac{1}{3}[16+4*54+192] = \frac{424}{3} = 141.3\bar{3}$

We determine the error bound using K defined by $f''''(x) = 24$, since this is a constant $K = 24$.

$E_S \leq \frac{24(4-0)}{180}(1)^4 = 0.53\bar{3}$. This is a good error bound for our integral! It will only get better with higher values of n.

The absolute error is $|140.8-141.3\bar{3}| = 0.53\bar{3}$, and the relative error is $\frac{0.53\bar{3}}{140.8} = 0.379\%$, which is a great approximation. ∎

Sidenote: Simpson's rule is exact for polynomials of degree 3 or less. In this case, we had a polynomial of degree 4, and so the error bound was exactly equal to the actual error. This is a unique thing, that only happens when the function is one degree higher than its exact. (For midpoint and trapezoidal, they are exact for a linear function, and the error bound is the same as the error for a quadratic.)
Also, all of these approximations behave better for polynomials because they are constructed using polynomials.

5.9 Improper Integrals

Improper integrals are integrals, $\int_a^b f(x)dx$, that cannot be evaluated using the Fundamental Theorem of Calculus as given because they do not satisfy the requirement that $f(x)$ is continuous on the closed interval $[a,b]$. There are two cases we will consider that do not meet this requirement:
1) Infinite bounds. Either a and/or b is infinite, so the closed interval does not exist because infinity is a construct and not a number.
2) A discontinuity of $f(x)$ is in the interval $[a,b]$.

5.9.1 Infinite Intervals

Cases, and how to deal with them:

If $\int_a^t f(x)dx$ exists for all values of $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$. If the limit exists and is finite, this integral is convergent. If the limit is infinite or does not exist, the integral is divergent.

Similarly, if $\int_t^b f(x)dx$ exists for all values of $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$. If the limit exists and is finite, this integral is convergent. If the limit is infinite or does not exist, the integral is divergent.

Combining the two, for an integral $\int_{-\infty}^\infty f(x)dx$, we split up the interval at any constant c. This $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$. We then evaluate both of these subintervals as above. If both limits exist and are finite, then the integral is convergent. If either of the limits is infinite or does not exist, then the original integral is divergent.

■ **Example 5.33** Evaluate $\int_1^\infty 2^{-x}dx$
We first replace the ∞ bound with a placeholder, like t $\lim_{t\to\infty} \int_1^t 2^{-x}dx$.
Recall: we can rewrite general exponentials $2^{-x} = e^{-x\ln(2)}$, so we can determine the antiderivative $F(x) = \dfrac{e^{-x\ln(2)}}{-\ln(2)}$.
Thus, $\lim_{t\to\infty} \dfrac{e^{-x\ln(2)}}{-\ln(2)}\Big|_1^t = \lim_{t\to\infty} \dfrac{e^{-t\ln(2)}}{-\ln(2)} - \dfrac{e^{-\ln(2)}}{-\ln(2)} = 0 - \dfrac{e^{-\ln(2)}}{-\ln(2)} = \dfrac{1}{2\ln(2)}$. ■

■ **Example 5.34** Evaluate $\int_{-\infty}^\infty e^{-2x}dx$
We need placeholders for both bounds in this case, so we will replace $-\infty$ with t, and ∞ with s.
We also need to split the interval at a constant (this is your choice), we will choose $c = 0$ for convenience.
$\int_{-\infty}^\infty e^{-2x}dx = \lim_{t\to-\infty} \int_t^0 e^{-2x}dx + \lim_{s\to\infty} \int_0^s e^{-2x}dx$
$\lim_{t\to-\infty} \dfrac{-1}{2} e^{-2x}|_t^0 + \lim_{s\to\infty} \dfrac{-1}{2} e^{-2x}|_0^s$
$\lim_{t\to-\infty} \dfrac{-1}{2}\left(e^0 - e^{-2t}\right) + \lim_{s\to\infty} \dfrac{-1}{2}\left(e^{-2s} - e^0\right) = \dfrac{-1}{2}(1-\infty) + \dfrac{-1}{2}(0-1)$, so the integral is Divergent because the limit is infinite. ■

5.9.2 Discontinuities

We also refer to this case as unbounded integrands. Function discontinuities, like vertical asymptotes, are often unbounded values of the function within the interval. These discontinuities can happen at an endpoint a or b, or at a value inside the interval (a,b).

Discontinuity at an endpoint:

We treat the discontinuity the same way we treated infinite bounds, replace it with a parameter and take the limit.
If $f(x)$ is continuous on $(a,b]$ and $\lim_{x\to a^+} f(x) \to \pm\infty$, then we rewrite $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$.
If $f(x)$ is continuous on $[a,b)$ and $\lim_{x\to b^-} f(x) \to \pm\infty$, then we rewrite $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$.

■ **Example 5.35** Evaluate $\int_1^5 \frac{1}{x-1}dx$

When $x=1$, $f(x) = \frac{1}{x-1}$ “blows up" to ∞, so we need to rewrite the integral
$\lim_{t\to 1^+} \int_t^5 \frac{1}{x-1}dx = \lim_{t\to 1^+} \ln|x-1||_t^5 = \lim_{t\to 1^+} \ln|5-1| - \ln|t-1| = \ln|4| - (-\infty)$, so the integral is Divergent. ■

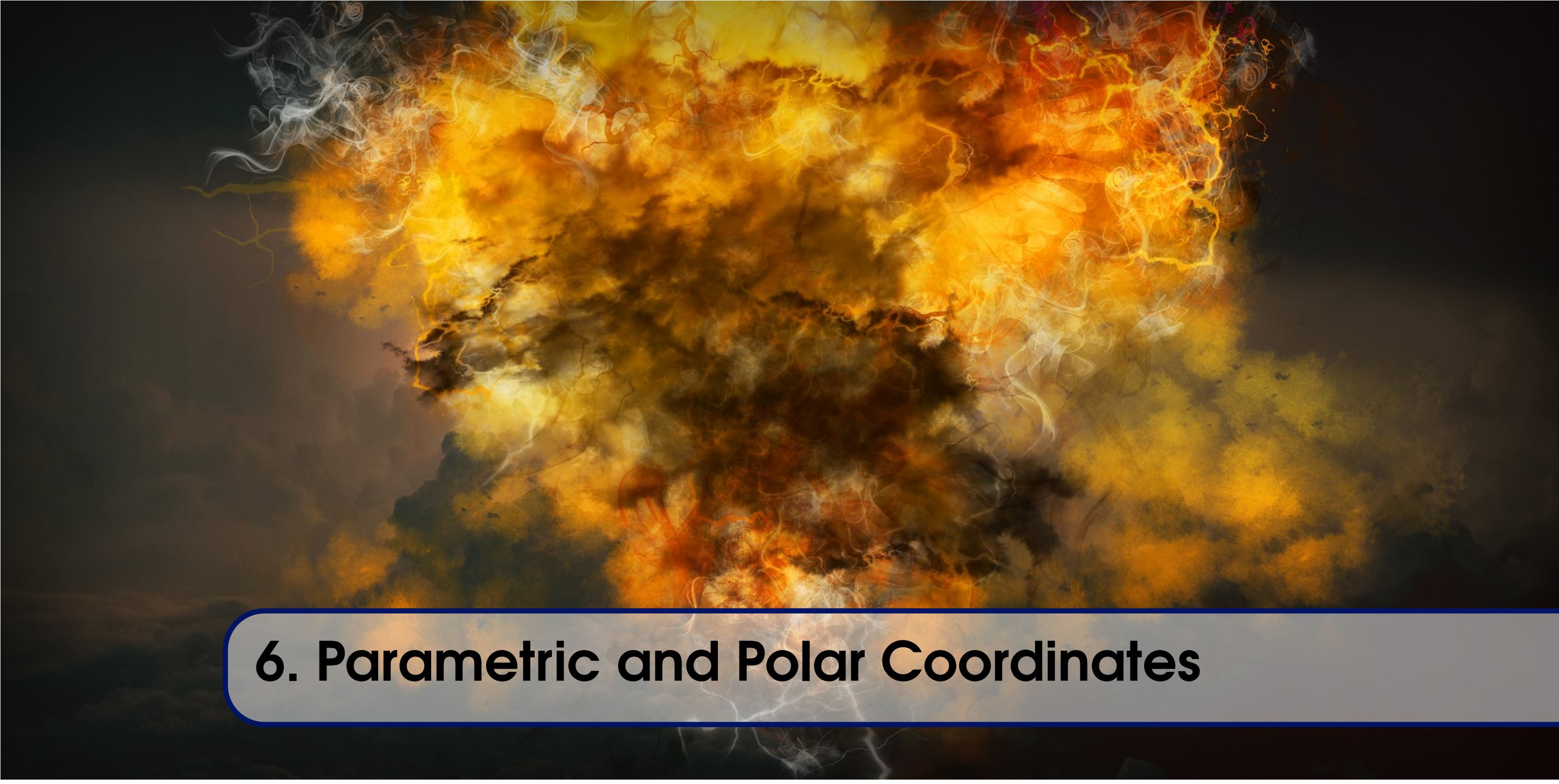
■ **Example 5.36** Evaluate $\int_0^1 \frac{1}{(x-1)^2}dx$

When $x=1$, $f(x) = \frac{1}{(x-1)^2}$ “blows up" to ∞, so we need to rewrite the integral
$\lim_{t\to 1^-} \int_0^t \frac{1}{(x-1)^2}dx = \lim_{t\to 1^-} \frac{-1}{x-1}|_0^t = \lim_{t\to 1^-} \frac{-1}{t-1} - \frac{-1}{0-1} \to \infty$, so the integral is Divergent.
■

Discontinuity inside the interval:
We treat these similarly to how we treated integrals with infinite bounds ($\int_{-\infty}^{\infty}$). Split them at the discontinuity, $\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx$, where p is the point of the discontinuity. Then, we solve the two integrals with the discontinuity on the endpoints.

■ **Example 5.37** Evaluate $\int_0^2 \frac{1}{\sqrt[3]{x-1}}dx$

$f(x) = \frac{1}{\sqrt[3]{x-1}}$ is discontinuous at $x=1$, so we split the interval at 1
$\int_0^2 \frac{1}{\sqrt[3]{x-1}}dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}dx$
Then, we replace the 1 with a placeholder, like t.
$\lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}dx + \lim_{t\to 1^+} \int_t^2 \frac{1}{\sqrt[3]{x-1}}dx$
$\lim_{t\to 1^-} \frac{3}{2}(x-1)^{2/3}|_0^t + \lim_{t\to 1^+} \frac{3}{2}(x-1)^{2/3}|_t^2$
$\lim_{t\to 1^-} \frac{3}{2}\left((t-1)^{2/3} - (0-1)^{2/3}\right) + \lim_{t\to 1^+} \frac{3}{2}\left((2-1)^{2/3} - (t-1)^{2/3}\right)$
$\frac{3}{2}(0-1) + \frac{3}{2}(1-0) = 0$, so the integral is Convergent. ■



6. Parametric and Polar Coordinates

6.1 Parametric Equations

Parametric equations are equations where redefine x and y in terms of a parameter, t. Then, rather than writing equations as $y = f(x)$, we write them through $x(t)$ and $y(t)$. These are briefly introduced in our Trigonometry course, and will be key at the beginning of Calculus III.

Since x and y are now defined in terms of t, we plot points $(x(t), y(t))$ in the $xy-$plane. So, to plot these points we pick values of t, and determine the points for x and y based on the values of t. The graph now also has an orientation, defined by arrows pointing along the curve for increasing values of t. Build a table:

t	x	y

■ **Example 6.1** Plot $x(t) = \cos(5t)$, $y(t) = 2\sin(5t)$ on $0 \leq t \leq \frac{\pi}{5}$
Then, algebraically determine an equation relating x and y alone.
To define the values of t that will be most helpful in the plot, we have to rely on our knowledge from trigonometry.
The unit circle values, $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, etc. are the values we know for $\sin(\theta)$ and $\cos(\theta)$. However, in this case we are evaluating $\sin(5t)$ and $\cos(5t)$. So, we seek t values where $5t =$ those known angles, and $0 \leq t \leq \frac{\pi}{5}$.

Angle $= 5t$	t
0	0
$\frac{\pi}{6}$	$\frac{\pi}{30}$
$\frac{\pi}{4}$	$\frac{\pi}{20}$
$\frac{\pi}{3}$	$\frac{\pi}{15}$
$\frac{\pi}{2}$	$\frac{\pi}{10}$
$\frac{2\pi}{3}$	$\frac{2\pi}{15}$
$\frac{3\pi}{4}$	$\frac{3\pi}{20}$
$\frac{5\pi}{6}$	$\frac{\pi}{6}$
π	$\frac{\pi}{5}$

We don't always need ALL of these points, but more points is absolutely helpful in depicting the picture.

Build the table:

t	x	y
0	1	0
$\frac{\pi}{30}$	$\frac{\sqrt{3}}{2}$	1
$\frac{\pi}{20}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{15}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{10}$	0	2
$\frac{2\pi}{15}$	$\frac{-1}{2}$	$\sqrt{3}$
$\frac{3\pi}{20}$	$\frac{-\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{6}$	$\frac{-\sqrt{3}}{2}$	1
$\frac{\pi}{5}$	-1	0

Use the points to sketch the graph:

You can find the function $y(x)$, generally, by solving for t in one equation, and substituting it into the other. However, this is not always a good idea, as it can lead to far more complicated results.

In this example, if we solve for t in the x equation we get $t = \frac{1}{5}\arccos(x)$. If we substitute this into $y(t)$, we have $y(x) = 2\sin(\arccos(x))$. This result as-is is not in a usable form, but if we use our knowledge of right triangles: treat $5t$ as θ in the triangle, then $x = \frac{\text{adjacent}}{\text{hypotenuse}}$, and so the adjacent side to the angle is x, and the hypotenuse is 1. We complete the triangle using they Pythagorean theorem, and find that the opposite side is $\sqrt{1-x^2}$. Then, $\sin(5t) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1}$, and so $y(x) = 2\sqrt{1-x^2}$. Which, if we use the knowledge from algebra, we might recognize that this is also representative of the ellipse $x^2 + \frac{y^2}{4} = 1$. However, that was a lot of work to arrive at an observation we can also make from the graph itself. Generally, when you see x and y defined by cos and sin, they will always result in some form of ellipse. ■

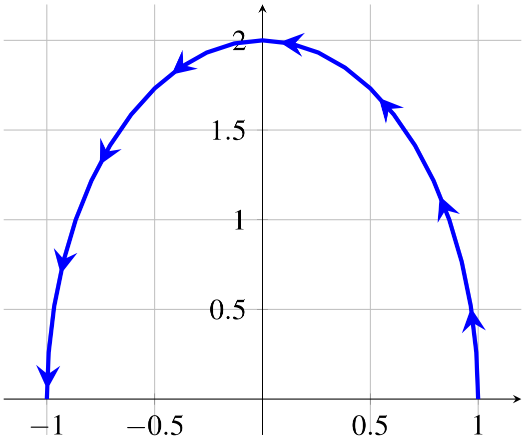


Figure 6.1: Depicted is the curve formed by the points in the table, which is the top half of an ellipse with a radius along x of 1 and a radius along y of 2. The arrows go along the curve in the direction of increasing t, which is counterclockwise.

■ **Example 6.2** Linear functions: Plot the parametric equation formed by $x = 2+t$ and $y = 3t-1$ on $0 \leq t \leq 2$. Then, determine the equation $y(x)$ which represents the same function.

Start with the table of values: We want to plot at least 5 points, so in this case we will take the interval $2-0$ and divide by 4 so that, including the endpoints, we will have 5 points. We will go from 0 to 2 in steps of $1/2$.

t	x	y
0	2	-1
$1/2$	$5/2$	$1/2$
1	3	2
$3/2$	$7/2$	$7/2$
2	4	5

Use the points to sketch the graph:

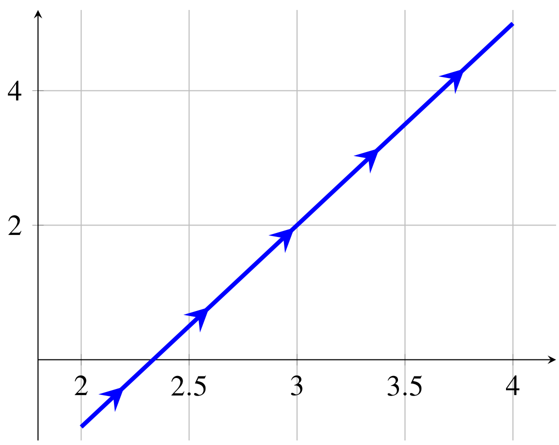


Figure 6.2: Depicted here is a line from $(2,-1)$ to $(4,5)$ formed by the parametric equations. The arrows go along the curve from lower left to upper right (from $(2,-1)$ to $(4,5)$).

In this problem, solving for t in the x equation is simpler because the function is linear already. $t = x-2$, so when we substitute this into $y(t)$, we get $y(x) = 3(x-2)-1 = 3x-7$. This line is the same as the parametric equations. ■

Summary: To plot, make a table and define the orientation with arrows pointing in the direction of positive t. If requested, substitute the functions appropriately to determine the function $y(x)$ (this is not always possible).

6.1.1 Calculus of Parametric Equations

This section focuses on the application of derivatives and integrals of parametric equations.

Recall: Chain rule $\frac{d}{dx}(f(g(x))) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x)$

Calculating the derivatives, $\frac{dy}{dx}$, from a parametric equation can be represented through a chain rule. When we solved for $y(x)$, we substituted $t(x)$ into $y(t)$ in order to calculate it. So, $\frac{dy}{dx} = \frac{d}{dx}(y(t(x))) = \frac{dy}{dt}(t(x)) \cdot \frac{dt}{dx}(x)$. Which, if we rewrite this, is actually equivalent to $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

If we refer back to our examples and calculate $\frac{dy}{dx}$ from our parametric equations, we can confirm this result.

From the first example: $\frac{dx}{dt} = -5\sin(5t)$, $\frac{dy}{dt} = 10\cos(5t)$, so $\frac{dy}{dx} = \frac{10\cos(5t)}{-5\sin(5t)} = -2\cot(5t)$. If we refer back to our triangle in this example, we can rewrite $\cot(5t) = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{\sqrt{1-x^2}}$, so $\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^2}}$.

Note that this method is easier than taking the derivative of the $y(x)$ function, and then replacing all the x values with t functions, at least in this case. $y(x) = 2\sqrt{1-x^2}$, so $\frac{dy}{dx} = 2 \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \frac{-2x}{\sqrt{1-x^2}}$. Then, $\frac{dy}{dx}(t) = \frac{-2\cos(5t)}{\sqrt{1-\cos^2(5t)}} = \frac{-2\cos(5t)}{\sin(5t)} = -2\cot(5t)$. Our answers are consistent, as they should be.

From the second example: $\frac{dx}{dt} = 1$, and $\frac{dy}{dt} = 3$, thus $\frac{dy}{dx} = \frac{3}{1} = 3$. Since this is a constant, there is no further conversion needed. This is also consistent with the $y(x) = 3x - 7$ function, as its derivative $\frac{dy}{dx} = 3$ also.

6.1.2 Tangent Lines using the Derivative

Once we calculate $\frac{dy}{dx}$, we now have the slope of the tangent line (recall that derivatives yield a rate of change in a function). Here, we combine those derivatives with our previous knowledge of tangent lines $T(x) = f'(a)(x-a) + f(a)$, to define the tangent line at a given point on a parametric curve.

■ **Example 6.3** Find the tangent line of $x = \cos(5t)$, $y(t) = 2\sin(5t)$ at $t = \frac{\pi}{20}$.

First, we already defined $\frac{dy}{dx} = -2\cot(5t)$ from the earlier section. We use the function in terms of

t because that is what we were given in the problem (a t value rather than an x value).
$5t = \frac{\pi}{4}$, so $\frac{dy}{dx} = -2\cot(\pi/4) = -2$. The slope of the tangent line at $t = \frac{\pi}{20}$ is -2.
Then, we use the parametric equations to define the point (x,y) we are building the tangent line from. $x = \cos(\pi/4) = \sqrt{2}/2$, $y = 2\sin(5t) = \sqrt{2}$, and so our point is $(\sqrt{2}/2, \sqrt{2})$.
We can recognize that in our tangent line equation, $a = \sqrt{2}/2$, and $f(a) = \sqrt{2}$, or we can just use point-slope form for a line: $y - y_0 = m(x - x_0)$. Either way, we have the tangent line: $y = -2(x - \sqrt{2}/2) + \sqrt{2}$. ■

6.2 Polar Coordinates

We define polar coordinates so that circles, and sections of circles, are defined by constant bounds. This transformation from Cartesian coordinates (x,y) to polar coordinates (r,θ) simplifies the integrals of functions that are circular. The polar coordinate system is defined with respect to the Cartesian coordinate system. All points, whether defined in Cartesian or polar, are plotted in the same $xy-$plane we have used since algebra. So, although the point has an ordered pair defined as (r,θ), the corresponding (x,y) point is the same point that we plot to represent the polar ordered pair.

In the definition of polar coordinates, $r =$ radius, or the distance from the origin (Recall from algebra: we used $\sqrt{x^2+y^2}$ to determine the distance of any point from the origin).

$\theta =$ the angle as measured with respect to the $x-$axis.

These both relate directly back to your unit circle in trigonometry. Recall: $x^2 + y^2 = r^2$, where r is the radius of the circle defined. Recall: $\tan(\theta) = \frac{y}{x}$, this is the same θ you used in your unit circle.

To convert from Cartesian to polar:
$r = \sqrt{x^2+y^2}$
$\theta = \arctan\left(\frac{y}{x}\right)$

To convert from polar to Cartesian:
$x = r\cos(\theta)$
$y = r\sin(\theta)$

A tool to help you remember these, aside from your unit circle, is the triangle

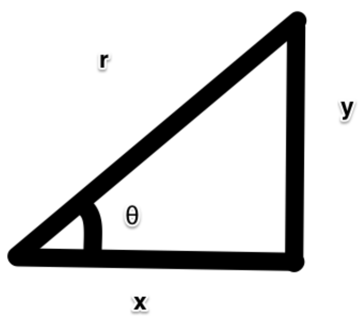


Figure 6.3: Depicted here is a triangle for defining polar coordinates in the $xy-$plane. It includes a right triangle where the hypotenuse is r, the vertical side is y, and the horizontal side is x. The angle formed between x and r is θ. It is depicted as though the point (x,y) is in the first quadrant of the $xy-$plane.

6.2.1 Plotting Points in Polar

There are two approaches to plotting points you are given in polar coordinates. The first, which you may want to start with, is to convert the points to Cartesian so that you know exactly what they are in the coordinates you are already familiar with. The second requires some confidence with the unit circle, and the meaning of r and θ physically. For this version, think of drawing the length r along the positive $x-$axis, and then rotating the line θ as you would in the unit circle. The end of the line corresponds to the point in the $xy-$plane. The second approach is faster, but requires some intuition - so practice if you wish to use it.

Additional cases: For negative r values, trace along the negative $x-$axis and rotate, if using the second approach. For negative θ values, rotate in the opposite direction (clockwise).

Plot the given points in polar coordinates (Note: in a problem, you would only be given the first column. The second column is work you may do to plot if you use the first approach.):

(r,θ)	(x,y)
$(3,\pi/4)$	$(3\sqrt{2}/2, 3\sqrt{2}/2)$
$(-1,\pi/3)$	$(-1/2,-\sqrt{3}/2)$
$(-2,\pi/2)$	$(0,-2)$
$(2,3\pi/2)$	$(0,-2)$

Note: the last two points in polar represent the same point in Cartesian. This is common, there are generally two points in polar that represent each point in Cartesian.

6.2.2 Polar Curves

Polar curves are represented by functions of r and/or θ.

Concept check: What is the graph defined by $r = 2$?
-A circle of radius 2
What is the graph of $\theta = \pi/4$?
-A line along the angle of $\pi/4$, which in Cartesian is $y = x$.

Polar curves are more generally defined through $r(\theta)$ functions. To plot these, make a table of at least 5 values for θ on the interval given (use $[0,2pi]$ if no interval is given), and determine the associated values for r. Then, plot the points, and connect them to form the curve.

■ **Example 6.4** Plot $r = \cos(2\theta)$

The number of points is important. If we only use 5 points on $[0,2\pi]$:

θ	r
0	1
$\pi/2$	-1
π	1
$3\pi/2$	-1
2π	1

This table is not particularly useful... How do these points connect? It honestly is not clear.

If we use 9 points instead:

θ	r
0	1
$\pi/4$	0
$\pi/2$	-1
$3\pi/4$	0
π	1
$5\pi/4$	0
$3\pi/2$	-1
$7\pi/4$	0
2π	1

The picture may become a bit clearer, and you may be able to now sketch the graph. However, more points will definitely make the shape of the graph clearer.

If we use 17 points:

θ	r
0	1
$\pi/8$	$\sqrt{2}/2$
$\pi/4$	0
$3\pi/8$	$-\sqrt{2}/2$
$\pi/2$	-1
$5\pi/8$	$-\sqrt{2}/2$
$3\pi/4$	0
$7\pi/8$	$\sqrt{2}/2$
π	1
$9\pi/8$	$\sqrt{2}/2$
$5\pi/4$	0
$11\pi/8$	$-\sqrt{2}/2$
$3\pi/2$	-1
$13\pi/8$	$-\sqrt{2}/2$
$7\pi/4$	0
$15\pi/8$	$\sqrt{2}/2$
2π	1

We now know exactly how the shape is built.

■

As we discussed in parametric equations, you can define the increments for these tables based on the function you are plotting. If we were asked to plot $r = \sin(3\theta)$, we would want to be aware that when $3\theta = 2\pi$, $\theta = \frac{2\pi}{3}$. So, the increments will have to be smaller than just using the standard $0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$, and 2π. For this function, we would actually want to use each of these values divided by 3, and then verify that we have enough points by checking the next point in the rotation, $0, \pi/12, \pi/6, 3\pi/12 = \pi/4, \pi/3, 5\pi/12, 3\pi/6 = \pi/2, 7\pi/12$, and $2\pi/3$. The next on in the list would be $9\pi/12$, and if it repeats a point we have already drawn - the graph is complete. If it gives us a new point - we may need to go all the way to 2π in increments of $3\theta = \pi/4$, so $\theta = \pi/12$.

This is why you want to get really comfortable with your unit circle again. Plotting polar curves really requires that you know the angles and values, and get comfortable with the points given by polar coordinates. Using $\pi/4$ increments (as we have done here), is a good starting point. However, there will be times that you may need to refine further and use the unit circle values from $\pi/6$ increments also. Review the unit circle, especially if it has been a while since you used it.

■ **Example 6.5** Plot $r = \sin(3\theta)$

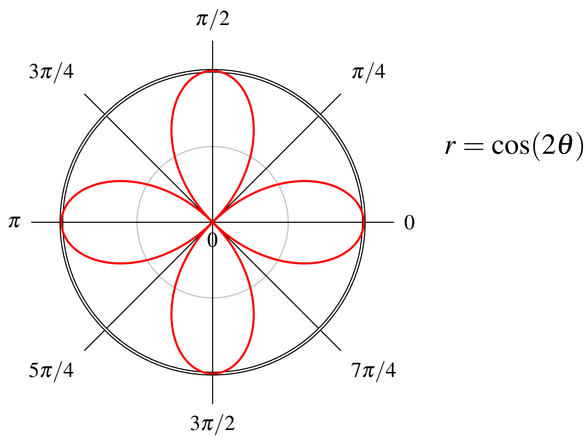


Figure 6.4: Depicted here is the graph of $\cos(2\theta)$ with circles drawn in for reference of the angles at $r = 0.5$ (in gray), and $r = 1$ (concentric black circles). The four-petal flower that is drawn rotates counter-clockwise from the Cartesian coordinates $(1,0)$, and flips across the origin at each $(0,0)$ crossing. There are lines drawn in for the x and y axes (labeled by their angles), and for each additional increment of $\pi/4$. This is to help with visualization of how this flower is drawn.

θ	r
0	0
$\pi/12$	$\sqrt{2}/2$
$\pi/6$	1
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	0
$5\pi/12$	$-\sqrt{2}/2$
$\pi/2$	-1
$7\pi/12$	$-\sqrt{2}/2$
$2\pi/3$	0
$3\pi/4$	$\sqrt{2}/2$
$5\pi/6$	1
$11\pi/12$	$\sqrt{2}/2$
π	0
$13\pi/12$	$-\sqrt{2}/2$

Points in black are the points we know are needed to sketch the graph. Points in blue are the next points in the rotation, and they correspond to new points of the graph. The point in red is the next point after the blue points, but it repeats the same point as $\pi/12$, and each point after it repeats the original points used. So, the red point, and all points after it are redundant and therefore are not used. ■

6.2.3 Sets of Points

Sets of points are generally a region of points that satisfy a relation. These sets are defined by a constraint on r or θ.

The line, $y = x$, is an example of a set of points that can be defined in (x,y) through $\{(x,y)|y = x\}$. This same line can be defined by $\{(r,\theta)|\theta = \pi/4\}$. This set then includes all values of r, with the specific angle $\theta = \pi/4$.

■ **Example 6.6** Sketch the set of points defined by $\{(r,\theta)|1 \leq r \leq 2\}$

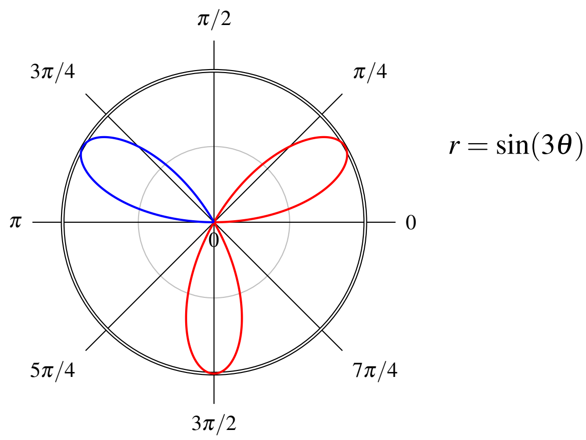


Figure 6.5: Depicted here is the graph of $\sin(3\theta)$ with circles drawn in for reference of the angles at $r = 0.5$ (in gray), and $r = 1$ (concentric black circles). The three-petal flower that is drawn rotates counter-clockwise from the Cartesian coordinates $(0,0)$ into the first quadrant, and flips across the origin at each $(0,0)$ crossing. There are lines drawn in for the x and y axes (labeled by their angles), and for each additional increment of $\pi/4$. This is to help with visualization of how this flower is drawn. The red curve includes all the points from $\theta = 0$ to $2\pi/3$, and the blue curve includes the points from $\theta = 2\pi/3$ to π. After $\theta = \pi$, the graph repeats over the petals shown.

Since there is no constraint on θ, all values of θ are included $0 \leq \theta \leq 2\pi$. We know that the graph of $r = 1$ is a circle of radius 1 from the last section. So, this set is all points between the circle of radius 1 and the circle of radius 2.

■

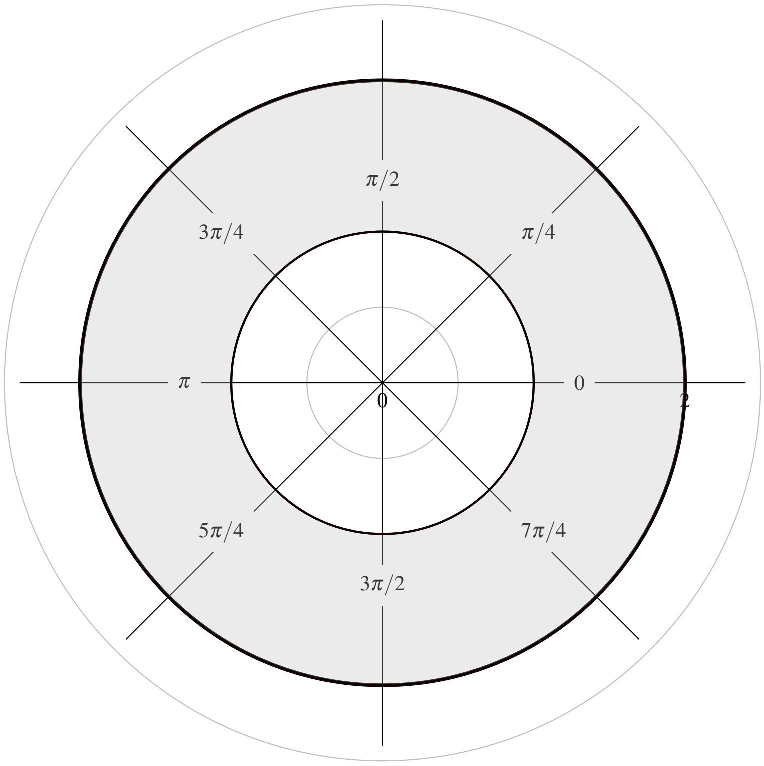


Figure 6.6: Shown here is a graph of two circles, centered at $(x,y) = (0,0)$. The smaller one has a radius of 1, and the larger one has a radius of 2. The region is then shaded gray between the two circles to represent the set of points defined in the problem.

6.3 Calculus in Polar Coordinates

6.3.1 Derivatives

We determine the derivatives in polar coordinates the same way we did for parametric equations. Assuming the form of polar equations, we define $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$.

Using the relationships $x = r\cos(\theta)$, and $y = r\sin(\theta)$, we can write this explicitly as:

$$\frac{\frac{d}{d\theta}(r\sin(\theta))}{\frac{d}{d\theta}(r\cos(\theta))} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

■ **Example 6.7** Find the slope, $\frac{dy}{dx}$, of $r = \cos(2\theta)$ at $\theta = \frac{\pi}{4}$.

We can simplify our computation by determining $\frac{dr}{d\theta} = -2\sin(2\theta)$ first.

Then, we can use the form of the derivative to define $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} = \frac{-2\sin(2\theta)\sin(\theta) + \cos(2\theta)\cos(\theta)}{-2\sin(2\theta)\cos(\theta) - \cos(2\theta)\sin(\theta)}$

To define the slope at $\theta = \frac{\pi}{4}$, we can then substitute the value directly:

$$\frac{-2\sin(\pi/2)\sin(\pi/4)+\cos(\pi/2)\cos(\pi/4)}{-2\sin(\pi/2)\cos(\pi/4)-\cos(\pi/2)\cos(\pi/4)} = \frac{-2(1)(\sqrt{2}/2)+0(\sqrt{2}/2)}{-2(1)(\sqrt{2}/2)-0(\sqrt{2}/2)} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1$$

If we refer back to the graph of the function, we can see that through $\theta = \pi/4$, the slope is approximately $\frac{dy}{dx} = 1$.

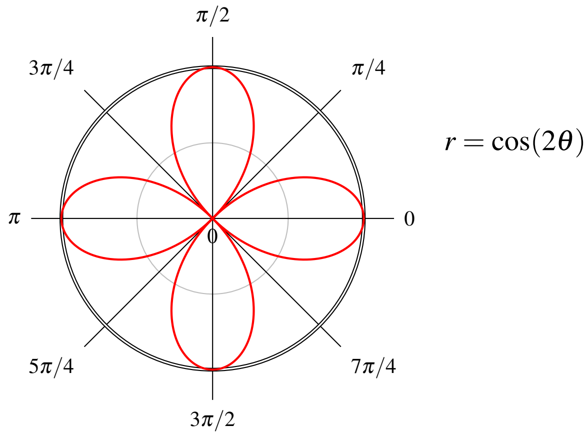


Figure 6.7: Depicted here is the graph of $\cos(2\theta)$ from earlier, and we use the line crossing through the curve at $\theta = \pi/4$ to reference the slope.

■

■ **Example 6.8** Determine the slope, $\frac{dy}{dx}$, for the graph $r = \theta$ at $\theta = \frac{\theta}{2}$.

$\frac{dr}{d\theta} = 1$

$\frac{dy}{dx} = \frac{(1)\sin(\theta)+\theta\cos(\theta)}{(1)\cos(\theta)-\theta\sin(\theta)}$

At $\theta = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{1+0}{0-\frac{\pi}{2}} = \frac{-2}{\pi}$ ■

6.3.2 Integrals

Integration in polar coordinates define the area of polar curves, and relate back to the area of a circle.

Recall: $A = \pi r^2$ is the area of a circle of radius r.

Since there are 2π radians in a circle, we can define a small segment of a circle through $\frac{\Delta\theta}{2\pi}$.

For r changing with respect to θ, polar curves $r(\theta)$, we can combine these pieces to define the integral for the area formed by a polar curve $\rightarrow \frac{r^2}{2}d\theta$. This is the term we integrate to determine the area bound by a polar curve.

$\int_\alpha^\beta \frac{1}{2}r^2 d\theta$

Note: This area is defined through the same ideas as the graph of a polar curve. Recall that r is the distance from the origin, so all area is now with respect to the origin, not with respect to the x-axis (as we were doing in (x,y)).

■ **Example 6.9** Determine the area swept our by $r=\cos(\theta)$ on $0\le\theta\le\frac{\pi}{2}$
$\int_0^{\pi/2}\frac{1}{2}(\cos(\theta))^2 d\theta = \frac{1}{2}\int_0^{\pi/2}\cos^2(\theta)d\theta$
This is a trigonometric integral, which we can simplify and solve by using the identity $\cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$
$\frac{1}{2}\int_0^{\pi/2}\frac{1}{2}(1+\cos(2\theta))\,d\theta=\frac{1}{4}\left(\theta+\frac{1}{2}\sin(2\theta)\right)|_0^{\pi/2}=\frac{1}{4}\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin(\pi)\right)-(0+0)\right]=\frac{\pi}{8}units^2$

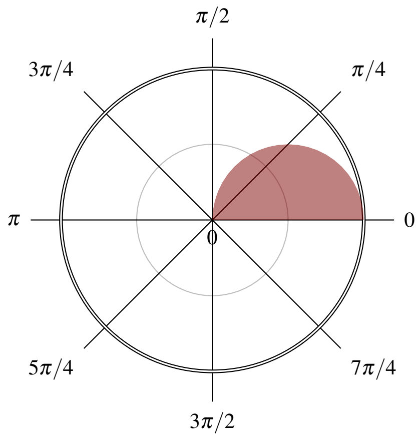


Figure 6.8: .

■

Area Between Two Polar Curves

We take the difference in the two areas, as we would normally. However, this looks a bit different because we have r_{outer}, the curve further from the origin, and r_{inner}, the curve closer to the origin.
$\int_\alpha^\beta\frac{1}{2}\left(r_{\text{outer}}^2-r_{\text{inner}}^2\right)d\theta$

■ **Example 6.10** Determine the area of the region inside both $r=1+\sin(\theta)$ and $r=1$.
The best way to determine your bounds is to sketch both curves, and then find the points of intersection.

Find the intersection points: $1+\sin(\theta)=1$, so $\sin(\theta)=0$. This happens when $\theta=0$, π, and 2π.

We see from the graph that on $0\le\theta\le\pi$, $r=1+\sin(\theta)$ is further away from the origin, and $r=1$ is the shape actually forming the area. On $\pi\le\theta\le 2\pi$, $r=1$ is further away from the origin, and $r=1+\sin(\theta)$ is the shape actually forming the area. So, the area inside both is found through:
$\int_0^\pi\frac{1}{2}1^2 d\theta+\int_\pi^{2\pi}\frac{1}{2}(1+\sin(\theta))^2 d\theta$
We expand this to: $\frac{1}{2}\int_0^\pi d\theta+\frac{1}{2}\int_\pi^{2\pi}\left(1+2\sin(\theta)+\sin^2(\theta)\right)d\theta$
We can evaluate most terms with an antiderivative, but $\sin^2(\theta)$ we will need to simplify using the identity $\sin^2(\theta)=\frac{1}{2}(1-\cos(2\theta))$.
$\frac{1}{2}\theta|_0^\pi+\frac{1}{2}\theta-\cos(\theta)|_\pi^{2\pi}+\int_\pi^{2\pi}\frac{1}{4}(1-\cos(2\theta))\,d\theta$

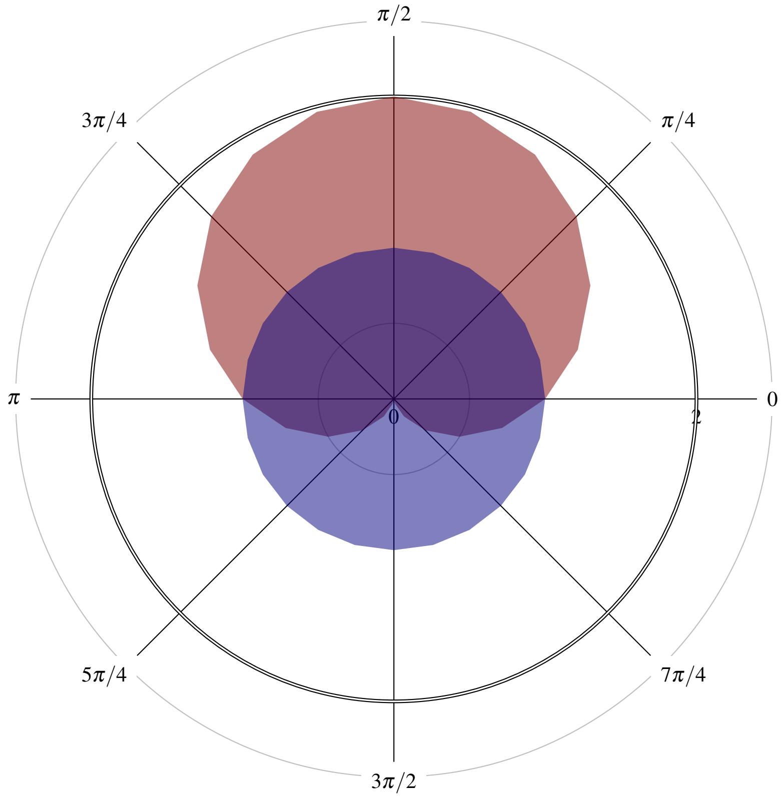


Figure 6.9: Shown here are shaded graphs of both polar curves, and so their overlapping area is darker by the overlap on $0 \leq \theta \leq 2\pi$. The darker region is the area we seek.

$$= \frac{\pi}{2} + \frac{\pi}{2} - (1+1) + \left(\frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)\right)\Big|_{\pi}^{2\pi}$$
$$= \pi - 2 + \frac{\pi}{4} - \frac{1}{8}(0-0) = \frac{5\pi}{4} - 2units^2$$ ■

■ **Example 6.11** For the same functions, determine the area between the two curves.
This time, we seek the areas of the red and blue regions, instead of the darker purple region in the previous plot.
We will use the same intersections, only now we will be taking the difference of our two functions.
Red region: $\int_0^{\pi} \frac{1}{2}\left((1+\sin(\theta))^2 - 1^2\right) d\theta$

$$\int_0^{\pi} \frac{1}{2}\left(2\sin(\theta) + \sin^2(\theta)\right) d\theta$$

In order to integrate the $\sin^2(\theta)$ term, we apply the identity $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$

$$\int_0^{\pi} \frac{1}{2}\left(2\sin(\theta) + \frac{1}{2}(1-\cos(2\theta))\right) d\theta$$
$$= \frac{1}{2}\left(-2\cos(\theta) + \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi} = \frac{1}{2}\left(2+2+\frac{\pi}{2} - \frac{1}{4}(0-0)\right) = \frac{1}{2}\left(4+\frac{\pi}{2}\right) = 2+\frac{\pi}{4}units^2$$

Blue region: $\int_{\pi}^{2\pi} \frac{1}{2}\left(1^2 - (1+\sin(\theta))^2\right) d\theta$

$$\int_{\pi}^{2\pi} \frac{1}{2}\left(1 - 1 - 2\sin(\theta) - \sin^2(\theta)\right) d\theta$$

We use the same identity to simplify, $\int_{\pi}^{2\pi} \frac{1}{2}\left(-2\sin(\theta) - \frac{1}{2}(1-\cos(2\theta))\right) d\theta$
$\frac{1}{2}\left(2\cos(\theta) - \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\right)|_{\pi}^{2\pi} = \frac{1}{2}\left((2+2) - \frac{1}{2}\left((2\pi-\pi) - \frac{1}{2}(0-0)\right)\right) = \frac{1}{2}\left(4 - \frac{1}{2}\pi\right) =$
$2 - \frac{\pi}{2} units^2$
■

Note the difference between these regions.

■ **Example 6.12** Find the area between $r = 3\sin(\theta)$ and $r = 3\cos(\theta)$.
First, we find the intersection points of the two curves: $3\sin(\theta) = 3\cos(\theta)$, there are technically two theta values where $\sin(\theta) = \cos(\theta)$, at $\theta = \frac{\pi}{4}$, and $\frac{5\pi}{4}$. However, if we graph the equations, only $\theta = \frac{\pi}{4}$ is used in this problem.

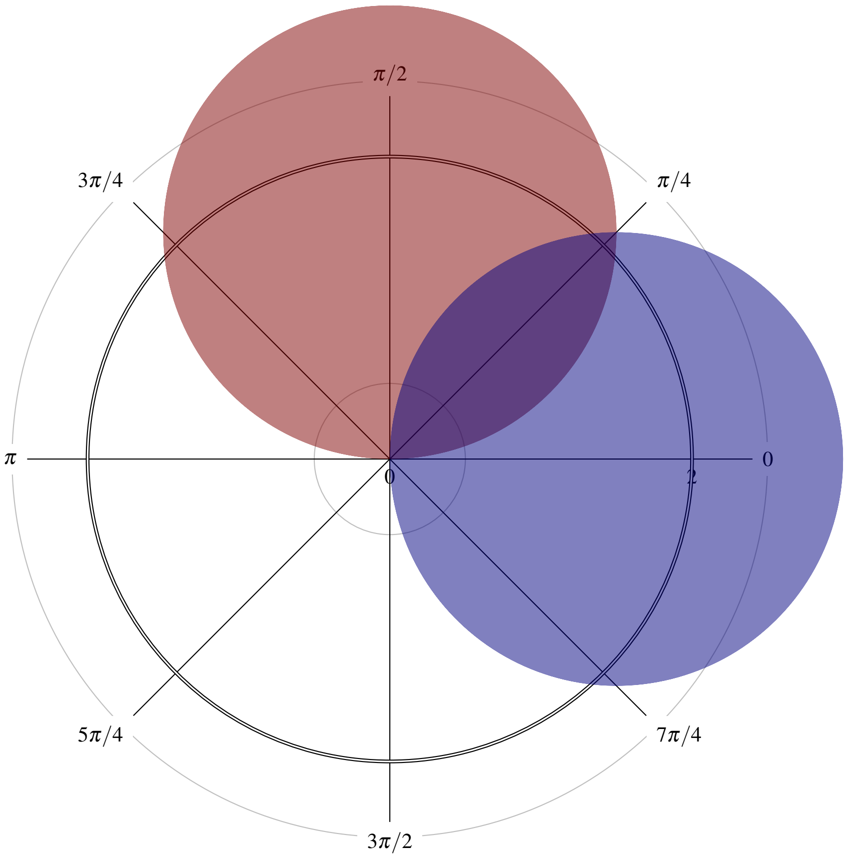


Figure 6.10: Shown here are shaded graphs of both polar curves, and so their overlapping area is darker by the overlap on $0 \leq \theta \leq \pi/2$. The darker region is the area we seek.

This is a rather unique case, because the area is actually split down the middle. One half comes from the $3\cos(\theta)$, and the other half comes from the $3\sin(\theta)$. Since our curves intersect at $\theta = \pi/4$ we can see that from $0 \leq \theta \leq \pi/4$ is inside the $3\sin(\theta)$ curve (area is measured from the origin to the curve). Then, from $\pi/4 \leq \theta \leq \pi/2$ is inside the $3\cos(\theta)$ curve.

So, the area inside both curves is found through:
$\int_{0}^{\pi/4} \frac{1}{2}(3\sin(\theta))^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(3\cos(\theta))^2 d\theta$

$\int_0^{\pi/4} \frac{9}{2} \sin^2(\theta) d\theta + \int_{\pi/4}^{\pi/2} \frac{9}{2} \cos^2(\theta) d\theta$

Applying our trigonometric identities to integrate: $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$, and $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$:

$\int_0^{\pi/4} \frac{9}{4}(1-\cos(2\theta))\, d\theta + \int_{\pi/4}^{\pi/2} \frac{9}{4}(1+\cos(2\theta))\, d\theta$

$\frac{9}{4}\left(\theta - \frac{1}{2}\sin(2\theta)\right)|_0^{\pi/4} + \frac{9}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)|_{\pi/4}^{\pi/2}$

$\frac{9}{4}\left((\pi/4-0) - \frac{1}{2}(1-0)\right) + \frac{9}{4}\left((\pi/2-\pi/4) + \frac{1}{2}(0-1)\right)$

$\frac{9}{4}\left(\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2}\right) = \frac{9}{4}\left(\frac{\pi}{2} - 1\right) units^2$

■

7. Sequences and Series

7.1 Sequences

Sequences are lists of numbers that satisfy a specific pattern or relation. Each term in the list is defined by its position in the list, which we denote by n (this is similar to the k in x_k used in Riemann sums).

Example form: $\{a_n\} = \{a_1, a_2, a_3, ...\}$ (note that sequences start with $n = 1$)

The more straight-forward format is when the sequences are defined by a function of n, $a_n = f(n)$

■ **Example 7.1** Define the first 5 terms of the sequence defined by $a_n = n^2 + 1$
$\{a_n\} = \{1^2 + 1, 2^2 + 1, 3^2 + 1, 4^2 + 1, 5^2 + 1, ...\} = \{2, 5, 10, 17, 26, ...\}$
■

The other format is when the sequences are defined by recurrence relations. These are situations where the current term is defined by a relation with the previous terms.

■ **Example 7.2** Define the first 5 terms of the sequence defined by $a_n = a_{n-1} + 2n - 1$, where $a_1 = 2$

$a_1 = 2$
$a_2 = 2 + 2*2 - 1 = 5$
$a_3 = 5 + 2*3 - 1 = 10$
$a_4 = 10 + 2*4 - 1 = 17$
$a_5 = 17 + 2*5 - 1 = 26$
$\{a_n\} = \{2, 5, 10, 17, 26, ...\}$
■

7.1.1 Limits of Sequences

When our limits are defined by a function, $f(n)$, we can compute the limit $\lim_{n\to\infty} a_n$ in exactly the same way we did in Calculus I with functions of x.

Recall: L'Hopital's rule, what causes a limit to diverge, and classification of a limit that converges vs. diverges.

For a recurrence relation, we can really only look at the relationship between terms. If our $a_n + 1$ is increasing, $a_{n+1} > a_n$, then our limit $\lim_{n\to\infty} a_n$ will go to ∞ and diverge.

For a recurrence relation, the value of $a_{n+1} - a_n$ must decrease as $n \to \infty$ in order for the limit to converge. So, we evaluate that difference to determine convergence or divergence of the limit.

■ **Example 7.3** Find the first 5 terms of $a_n = \frac{1}{3^n}$, and determine if the sequence converges
$\{a_n\} = \left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \dots\right\}$
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3^n} \to \frac{1}{\infty} \to 0$, so the limit Converges, and the sequence Converges. ■

■ **Example 7.4** Find the first 5 terms of $a_n = \frac{1}{2}a_n$, $a_1 = 1$, and then determine if the sequence converges.
$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$
$\lim_{n\to\infty} a_n =$? Recall above, for recurrence relations we need to show that $a_{n+1} - a_n$ decreases as $n \to \infty$. Since $a_{n+1} = \frac{1}{2}a_n$, $a_{n+1} - a_n = \frac{-1}{2}a_n$, and $a_n = \frac{1}{2}a_{n-1} = \frac{1}{4}a_{n-2} = \frac{1}{2^{n-1}}a_1$, so we can clearly see that $a_n \to 0$ as $n \to \infty$, which is going to show that the sequence is convergent.
This is actually a special relation where this recurrence relation is equivalent to the function notation form $a_n = \frac{1}{2^{n-1}}$, and we know that $\lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$ so the sequence is convergent. ■

■ **Example 7.5** Determine the $\lim_{n\to\infty} a_n$ for $a_n = \frac{4n^2}{n^2-1}$
Since this sequence is defined as a function of n, we can treat the limit in the same way we did a limit of $f(x)$ with respect to x.
$\lim_{n\to\infty} \frac{4n^2}{n^2-1} \to \frac{\infty}{\infty}$ Recall: L'Hopital's rule, which we use to evaluate indeterminate limits. In such a case, we can evaluate the same limit as a ratio of the derivatives of our numerator and denominator.
$\lim_{n\to\infty} \frac{8n}{2n} \to \frac{\infty}{\infty} \to \lim_{n\to\infty} \frac{8}{2} = \frac{8}{2} = 4$ Since this is a defined constant, we say the limit converges, and therefore the sequence also converges. ■

■ **Example 7.6** Determine the $\lim n \to \infty a_n$ for $a_n = \frac{2e^n}{e^n+1}$
We evaluate the limit $\lim_{n\to\infty} \frac{2e^n}{e^n+1} \to \frac{\infty}{\infty}$
We again apply L'Hopital's rule in order to evaluate the limit.
$\lim_{n\to\infty} \frac{2e^n}{e^n}$ This reduces to $\frac{2}{1}$ generally, and so $\lim_{n\to\infty} \frac{2}{1} = \frac{2}{1} = 2$. Since this is a defined constant, the limit converges, and therefore the sequence also converges. ■

7.1.2 Defining the Relation of a Sequence

Pattern recognition: Determining the relation that forms a sequence is entirely dependent on pattern recognition. Humans are actually quite good at pattern recognition, but when put in the context of a mathematical sequence - you may need to actively work to build the connections to find the patterns.

■ **Example 7.7** Given the sequence $\left\{1, \frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \ldots\right\}$
Find the next two terms, then write the relation as a function and a recurrence relation.
We are looking for patterns, the first one we are likely to notice is that this sequence alternates between positive and negative values. How do we make that happen in a function?
We use the knowledge that two negative values multiplied together make a positive, then another negative will make the function negative again: $(-1)(-1)(-1) = -1$. We generalize this with a $(-1)^n$ or $(-1)^{n-1}$, depending on which terms of the sequence are negative.
In this case, the first term, when $n = 1$ is positive, so we will use the second notation $(-1)^{n-1}$ (note that $(-1)^{n+1}$ is also fine to use here). This will force make the even values of n result in a multiple of -1.
The second pattern is that we divide by 2 from each term to the next. So, $a_1 = \frac{1}{2^0}$, $a_2 = \frac{-1}{2^1}$, $a_3 = \frac{1}{2^2}$, and $a_4 = \frac{-1}{2^3}$
Combining these two pieces, we can define $a_5 = \frac{1}{2^4}$ and $a_6 = \frac{-1}{2^5}$
The function form for this sequence is then generalized to $a_n = \frac{(-1)^{n-1}}{2^{n-1}} = \left(\frac{-1}{2}\right)^{n-1}$.
Since each term is multiplied by $\frac{-1}{2}$ to get the next term, the recurrence relation is $a_{n+1} = \frac{-1}{2}a_n$, where $a_1 = 1$.
Is this a convergent sequence?
$\lim_{n\to\infty}\left(\frac{-1}{2}\right)^{n-1} = 0$, so the sequence converges. ■

■ **Example 7.8** Given the recursion relation, $a_{n+1} = \sqrt{1+a_n}$, $a_1 = 1$
Define the first 5 terms of the sequence, and determine if the sequence is converging or diverging.
$\{a_n\} = \left\{1, \sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1\sqrt{2}}}}, \ldots\right\}$
These values numerically are approximately $\{1, 1.41, 1.55, 1.598, 1.61, \ldots\}$
Since this is a recurrence relation, we cannot simply take the limit as $n \to \infty$, so we have to look at the values $a_{n+1} - a_n$: $\{0.41, 0.14, 0.048, 0.012, \ldots\}$
From this, we can see that $a_{n+1} - a_n$ is decreasing as n increases. Thus, we can conclude that the sequence is converging, to approximately 1.62. ■

7.1.3 Vocabulary

Monotonic: the sequence only behaves in one way. It either increases, for all values of n as n increases. Or, it only decreases, for all values of n as n increases.

A sequence is monotonically increasing if $a_{n+1} > a_n$ for all values of n.
A sequence is monotonically decreasing if $a_{n+1} < a_n$ for all values of n.

Nonincreasing: the sequence does not increase. $a_{n+1} \leq a_n$ for all values of n.
Nondecreasing: the sequence does not decrease. $a_{n+1} \geq a_n$ for all values of n.

Bounded: all terms of the sequence are defined. (None are infinite)

7.1.4 Analysis of Sequences

Using the vocabulary terms given above, define properties of a given sequence.

■ **Example 7.9** Define the properties of the sequence formed by $a_n = \frac{4n^2}{n^2-1}$
To determine the properties, we look at the next term and compare it to the current term.
$a_{n+1} = \frac{4(n+1)^2}{(n+1)^2-1} = \frac{4n^2+8n+4}{n^2+2n}$
Now, we want to determine the relationship between a_{n+1} and a_n ($<$, $>$, $\leq$, $\geq$)
$\frac{4n^2+8n+4}{n^2+2n} ?? \frac{4n^2}{n^2-1}$
Since $n \geq 1$, both denominators are non-negative, so if we cross-multiply the relation should remain consistent.
$(4n^2+8n+4)(n^2-1) ?? 4n^2(n^2+2n)$
$4n^4+8n^3+4n^2-4n^2-8n-4 ?? 4n^4+8n^3$
Simplifying, we see $-8n-4 ?? 0$, and since $n \geq 1$, the $?? =<$ since the left side must always be less than 0.
Then, $a_{n+1} < a_n$ for all n. We call this a monotonically decreasing sequence, which is also a nonincreasing sequence. ■

■ **Example 7.10** Define the properties of the sequence formed by $a_n = \frac{2e^n}{e^n+1}$
$a_{n+1} = \frac{2e^{n+1}}{e^{n+1}+1}$
$\frac{2e^{n+1}}{e^{n+1}+1} ?? \frac{2e^n}{e^n+1}$
Since our denominators are positive for all n, we can again cross-multiply
$2e^{n+1}(e^n+1) ?? 2e^n(e^{n+1}+1)$
$2e^{2n+1}+2e^{n+1} ?? 2e^{2n+1}+2e^n$
Simplifying we have $2e^{n+1} ?? 2e^n$
$e ?? 1$
$e > 1$, so $a_{n+1} > a_n$ for all n.
This is a monotonically increasing sequence, which is also a nondecreasing sequence. ■

7.1.5 Geometric Sequences

Geometric sequences are a particular form that has a specific relation for convergence and divergence. They are sequences that have known properties.

The general form is represented through $a_n = Cr^n$, where C is the coefficient, r is the ratio, and n is the term of the sequence.

When $|r| < 1$, the geometric sequence will decrease toward 0 and converge.
When $|r| > 1$, the geometric sequence will increase toward ∞ and diverge.

For negative r values, the sequence oscillates, but the statements above still hold.

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r| < 1 \\ 1, & r = 1 \\ DNE, & |r| > 1 \text{ or } r = -1 \end{cases}$$

■ **Example 7.11** Determine whether the sequence converges or diverges, and whether it does so monotonically, or by oscillation.

1. $\lim_{n\to\infty}(-0.5)^n \to 0$ this limit converges, so the sequence also converges. Since r is negative (-0.5), the sequence converges by oscillation.
2. $\lim_{n\to\infty}(1.2)^n \to \infty$, this limit diverges, so the sequence also diverges. Since the $r = 1.2$ is positive, the sequence diverges monotonically.
3. $\lim_{n\to\infty} 4\left(\frac{1}{3}\right)^n \to 0$, this limit converges, so the sequence also converges. Since $r = \frac{1}{3}$ is positive, the sequence converges monotonically.
4. $\lim_{n\to\infty} \frac{1}{2}(-3)^n \to \infty$, this limit diverges, so the sequence also diverges. Since $r = -3$ is negative, the sequence diverges by oscillation.

■

7.1.6 The Squeeze Theorem

The Squeeze Theorem in Calculus I was used to bound limit values to determine if a limit converged or diverged. We use the same concept here to determine if the sequence converges or diverges.

Given three sequences, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, where $\{a_n\}$ and $\{c_n\}$ are known, and $a_n \leq b_n \leq c_n$ for all n.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then we know that $\lim_{n\to\infty} b_n = L$ also, by the Squeeze Theorem.

■ **Example 7.12** Find the limit, $\lim_{n\to\infty} b_n$, for $b_n = \frac{\sin(n)}{2n-1}$ by applying the Squeeze Theorem
To apply the Squeeze Theorem, we need to use the fact that $sin(n)$ is a bounded function (it never goes above 1, and never goes below -1.
This allows us to write that $-1 \leq \sin(n) \leq 1$, so $\frac{-1}{2n-1} \leq \frac{\sin(n)}{2n-1} \leq \frac{1}{2n-1}$
We then evaluate the limits of the bounds: $\lim_{n\to\infty} \frac{-1}{2n-1} = 0$, and $\lim_{n\to\infty} \frac{1}{2n-1} = 0$
Since the two limits are BOTH $= 0$, we can then say that $\lim_{n\to\infty} \frac{\sin(n)}{2n-1} = 0$ also, by the Squeeze Theorem. ■

Theorem 7.1.1 A bounded, monotonic sequence converges.

This is true because for a function to be bounded, it must not be undefined for any value. So, the sequence will always approach a defined value, which will result in a convergent sequence.

7.1.7 Application: Drug Doses

We can model discrete operations, like drug doses, using recurrence relations.

■ **Example 7.13** If you take 50mg of Vitamin C, every 8 hours, and it has a half life of 4 hours, define a recurrence relation for the mg in your system after each dose.
First note: since a new dose is taken every 8 hours, it makes sense to model it so each term in the

sequence is for 8 hour-increments. Then, if the half-life is only 4 hours, it is applied twice in 8 hours, resulting in $\frac{1}{4}$ the previous dose remaining in your system.
Putting these pieces together, we can write the terms through $a_{n+1} = \frac{1}{4}a_n + 50$.

Will this converge?
$\{a_n\} = \{50, 62.5, 65.625, 66.40625, 66.6016, ...\}$
Since this is a recurrence relation, we look at $a_{n+1} - a_n$: $\{2.5, 3.125, 0.78125, 0.19535, ...\}$
We can see that this value decreases, as n increases, so the sequence appears to converge, roughly to $67mg$.
■

7.1.8 Convergence and Growth Rates

Given two sequences, $\{a_n\}$ and $\{b_n\}$, we can define their growth (or convergence) rates through the ratio of the two sequences.

When $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and both sequences are divergent, then we can say that b_n grows faster than a_n.

When $\lim_{n\to\infty} \frac{a_n}{b_n} \to \infty$, and both sequences are divergent, then we can say that a_n grows faster than b_n.

The same conclusions can be reached if both sequences are convergent.
For example, if a_n is convergent, and b_n is divergent, then $\lim_{n\to\infty} \frac{a_n}{b_n} \to 0$ because the divergent sequence is (of course) growing faster than the convergent sequence.

This allows us to define the relationship between two sequences.

7.2 Infinite Series and Partial Sums

We have used series notation in Riemann Sums and Numerical Integration, and in both of those cases we worked with finite series corresponding to the specific problem being solved. An infinite series will take the same form, only instead of stopping after a specified N, the upper bound on the sigma notation for the sum will be ∞.

The terms are formed similarly to sequences, but instead of forming a list, we will be adding the terms together to form the result of the series.

$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + ... + a_n + ...$ (continues on infinitely)

Partial Sums

We stop after some specified N terms in the series to determine a partial sum. We add a specified number of terms, rather than seeking the infinite series. These partial sums can help us to determine the value of the infinite series.

■ **Example 7.14** Find the first 3 partial sums of $\sum_{k=1}^{\infty} 2^k$, and determine if the series converges or diverges.
The first partial sum is denoted $S_1 = \sum_{k=1}^{1} 2^k = 2^1 = 2$

The second partial sum is $S_2 = \sum_{k=1}^{2} 2^k = 2^1 + 2^2 = 6$
The third partial sum is $S_3 = \sum_{k=1}^{3} 2^k = 2^1 + 2^2 + 2^3 = 14$
Generally, the nth partial sum is $S_n = \sum_{k=1}^{n} 2^1 + 2^2 + 2^3 + ... + 2^n$
If we evaluate $\lim_{n\to\infty} S_n \to \infty$ the limit diverges, the sequence of terms diverges, and since they are all added together, the series also diverges.
For future reference, if this limit diverges, the series diverges. (The converse is not true for series)
■

If the sequence of partial sums has a limit L, $\lim_{n\to\infty} S_n = L$, the infinite series converges. i.e., If $\lim_{n\to\infty} S_n = L$, then $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k = L$.

7.2.1 Types of Infinite Series

These are specific forms of infinite series with known convergence properties.

Geometric Series

Similar to our Geometric Sequences, they are of the form $\sum_{k=0}^{n-1} ar^k = a + ar + ar^2 + ... + ar^{n-1}$. Note: These sequences start at $k = 0$, not $k = 1$. a is the coefficient (does not change), and r is the ratio between terms (is multiplied with each term).

■ **Example 7.15** Determine the partial sum, S_4, for $\sum_{k=0}^{3} 0.2(0.5)^k$.
First, we can notice that $a = 0.2$, $r = 0.5$, and there are 4 terms so $n = 4$, S_4.
If we write out the sum, $S_4 = 0.2\,(1 + 0.5 + 0.25 + 0.125) = 0.375$
■

For geometric sums, we actually can define any partial sum based on the coefficient and the ratio.
$S_n = a\dfrac{1-r^n}{1-r}$

We can verify this is the same as the last example: $S_4 = 0.2\dfrac{1-(0.5)^4}{1-0.5} = 0.4\left(1 - \dfrac{1}{16}\right) = \dfrac{4}{10}\dfrac{15}{16} = \dfrac{15}{40} = \dfrac{3}{8} = 0.375$✓

For the infinite series, we can look at the behavior of the partial sums as $n \to \infty$
$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} ar^k$

$$\text{Recall: } \lim_{n\to\infty} r^n = \begin{cases} 0 & |r| < 1 \\ 1 & r = 1 \\ DNE & |r| > 1 \text{ or } r = -1 \end{cases}$$

So, $\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} a\dfrac{1-r^n}{1-r} = a\dfrac{1}{1-r}$, if $|r| < 1$
It DNE if $|r| \geq 1$

For the series from our example, $\lim_{n\to\infty} a\dfrac{1-r^n}{1-r} = \lim_{n\to\infty} 0.2\dfrac{1-(0.5)^n}{1-0.5} = 0.2\dfrac{1}{0.5} = 0.4$
So, that infinite series converges to 0.4.

■ **Example 7.16** Determine the partial sums S_5 and S_{27}, and the sum of the full series (if it converges), for $\sum_{k=0}^{\infty} 0.1(0.4)^k$

$S_5 = 0.1\frac{1-(0.4)^5}{1-0.4} = 0.1\frac{1-0.01024}{0.6} = 0.16496$

$S_{27} = 0.1\frac{1-(0.4)^{27}}{1-0.4} = 0.1\frac{1-1.8\times 10^{-11}}{0.6} \approx 0.1667$

$\sum_{k=0}^{\infty} 0.1(0.4)^k = \lim_{n\to\infty} a\frac{1-(0.4)^n}{1-0.4} = 0.1\frac{1}{0.6} = \frac{1}{6} \approx 0.1667$

■

■ **Example 7.17** Determine if $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}}$ converges or diverges, and if it converges determine its value.

This is a geometric series, but it's a little hidden... $\frac{3^k}{4^k} = \left(\frac{3}{4}\right)^k$

So, $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^k$, so $a = 0.25$ and $r = 0.75 < 1$, so this series will converge according to the geometric series test.

Using the relation above, $\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}} = \lim_{n\to\infty} \frac{1}{4}\frac{1-(0.75)^n}{1-0.75} = \frac{0.25}{0.25} = 1$ So, the infinite series converges to 1. ■

Telescoping Series

Telescoping series are a special form where the middle terms cancel each other out, as more terms are added to the series. So, when the infinite series is taken, only the first and last terms remain. These are often formed by a difference of two quotients.

■ **Example 7.18** Determine the value of the telescoping series $\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$

If we write this series out, we find: $\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{16}\right) + \left(\frac{1}{16} - \frac{1}{32}\right) + \ldots$

As this continues, all the middle terms cancel out. So, all that remains are the first term, $\frac{1}{2}$, and the last term $\lim_{n\to\infty} \frac{-1}{2^{n+1}} = 0$.

So, then $\sum_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \frac{1}{2}$, the series is convergent. ■

We can arrive at this form from partial fractions, recall: partial fraction decomposition.

■ **Example 7.19** Determine the value of $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$

Since we do not have a technique for fractions, we can split the fraction to get a telescoping series.

$\frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2}$

$1 = Ak + 2A + Bk$, so $1 = 2A$ and $0 = A + B$

$A = \frac{1}{2}, B = \frac{-1}{2}$.

So, $\frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$

$\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots\right]$

Again, we see all the middle terms cancel, but this time we keep the first part of the first two terms of the series. Note: the denominators differ by 2, so we keep 2 terms, all others cancel.

$\lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}(0-0)$, so only the terms listed above are included.
$\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$ ■

7.3 The Divergence and Integral Tests

Using tools we have worked with thus far in the semester, we can define the first two general tests to use on infinite series when we want to determine their convergence.

7.3.1 The Divergence Test

The divergence test ONLY tests for divergence. You cannot use it to say anything about convergence. It is a simple, quick test to determine if a series diverges. But, if it passes the test, the series may still diverge.

We know that for a series to converge, $\lim_{k\to\infty} a_k = 0$.
Thus, if $\lim_{k\to\infty} a_k \neq 0$, the series must diverge.

If $\lim_{k\to\infty} a_k = 0$, then you need to use another test, because the divergence test has told you NOTHING about whether the series converges or diverges.

This is a nice first test, because it is quick, and if the limit is not zero - you are done.

■ **Example 7.20** Apply the divergence test to the series $\sum_{k=1}^{\infty} \frac{k-1}{k}$
$\lim_{k\to\infty} \frac{k-1}{k} \to \frac{\infty}{\infty}$ We can apply L'Hopital's rule, to get $\lim_{k\to\infty} \frac{1}{1} = 1 \neq 0$, so the series Diverges.
Therefore, $\sum_{k=1}^{\infty} \frac{k-1}{k}$ diverges by the divergence test. ■

■ **Example 7.21** Apply the divergence test to the series $\sum_{k=1}^{\infty} \frac{2^k+1}{4^k}$
$\lim_{k\to\infty} \frac{2^k+1}{4^k} = \lim_{k\to\infty} \left(\frac{2}{4}\right)^k + \left(\frac{1}{4}\right)^k = 0+0 = 0$ the divergence test is INCONCLUSIVE.
Another test needs to be implemented to determine convergence or divergence. ■

7.3.2 The Integral Test

The integral test can be applied to a positive, decreasing, continuous function $a_k = f(k)$ on $[1,\infty)$. We use the test to determine convergence of an infinite series, $\sum_{k=1}^{\infty} a_k$, by evaluating the equivalent improper integral, $\int_1^{\infty} f(k)dk$. The series and the integral will either both converge, or both diverge. So, we can use the integral test to determine convergence, but we cannot use it to determine the value of the infinite series.

The Harmonic Series

The Harmonic series is a special series that we will use as a reference for later series.

$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

If we apply the Divergence test, $\lim_{k\to\infty} \frac{1}{k} = 0$, so the test is inconclusive.

However, we can apply the Integral test to show that the limit actually diverges.

■ **Example 7.22** Determine if the Integral test can be applied to $\sum_{k=1}^{\infty} \frac{1}{k}$. If so, apply the Integral test to show that the Harmonic series diverges.
In order to apply the Integral test, $\frac{1}{k}$ must be a positive, decreasing, continuous function on $[1,\infty)$.
Is $\frac{1}{k}$ positive for all k on $[1,\infty)$? Yes.
Is $\frac{1}{k+1} < \frac{1}{k}$ for all k on $[1,\infty)$? Yes.
Is $\frac{1}{k}$ continuous on $[1,\infty)$? Yes.
We can apply the integral test!
We evaluate the improper integral $\int_1^{\infty} \frac{1}{k} dk = \lim_{t\to\infty} \int_1^t \frac{1}{k} dk = \lim_{t\to\infty} \ln|k| - \ln|1| \to \infty$, so the improper integral diverges.
Therefore, by the integral test, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ also diverges. ■

■ **Example 7.23** Determine if the Integral test can be applied to $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$. If so, apply the Integral test to determine convergence.
Is $\frac{1}{k^2+1}$ positive for all k on $[1,\infty)$? Yes.
Is $\frac{1}{(k+1)^2+1} < \frac{1}{k^2+1}$ for all k on $[1,\infty)$? Yes.
Is $\frac{1}{k^2+1}$ continuous on $[1,\infty)$? Yes.
We can apply the Integral test!
Evaluate $\int_1^{\infty} \frac{1}{k^2+1} dk = \lim_{t\to\infty} \int_1^t \frac{1}{k^2+1} dk = \lim_{t\to\infty} \arctan(t) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, so the integral converges.
Therefore, by the integral test, the series $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ will also converge. ■

7.3.3 P-Series Test

The p-series test is a nice test for any series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$. It defines convergence based on the value of p. We learned from the Harmonic series that when $p = 1$ the series diverges, so for which values of p does the series converge?

We can look at the general form, using the integral test. $\sum_{k=1}^{\infty} \frac{1}{k^p}$.
$\frac{1}{k^p}$ is positive for any value of p
$\frac{1}{k^p}$ is decreasing only for $p > 0$
$\frac{1}{k^p}$ is continuous on $[1,\infty)$ for all p
So, we can only apply the integral test for positive values of p.

$$\int_1^{\infty} \frac{1}{k^p} dk = \lim_{t\to\infty} \int_1^t \frac{1}{k^p} dk = \lim_{t\to\infty} \int_1^t k^{-p} dk = \lim_{t\to\infty} \begin{cases} \frac{k^{-p+1}}{-p+1} & p \neq 1 \\ \ln|k| & p = 1 \end{cases} \Big|_1^t$$

We already know for the case when $p = 1$, this diverges.

When $p > 1$, $\lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} = -\frac{1}{1-p}$ and converges.
When $p < 1$, $\lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \to \infty - \frac{1}{1-p}$ and diverges.

So, through the p-series test, we can define that the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$, and diverges for $p \leq 1$.

■ **Example 7.24** Use the p-series test to determine the convergence of $\sum_{k=1}^{\infty} k^{-3/2}$.
$k^{-3/2} = \frac{1}{k^{3/2}}$, which is the form of a p-series, with $p = 3/2$.
$3/2 > 1$, therefore this series will converge by the p-series test.

Aside: We can also show it is convergent by the integral test directly, but this is not necessary. Since, $k^{-3/2}$ is a positive, decreasing, continuous function on $[1,\infty)$, the series will converge if the integral $\int_1^{\infty} k^{-3/2}dk$ converges.
$\lim_{t\to\infty} \int_1^t k^{-3/2}dk = \lim_{t\to\infty} -2t^{-1/2} + 2 = 2$ converges!
Therefore, by the integral test, the series $\sum_{k=1}^{\infty} k^{-3/2}$ also converges. ■

■ **Example 7.25** Use the p-series test to determine the convergence of $\sum_{k=1}^{\infty} k^{-1/2}$
$k^{-1/2} = \frac{1}{k^{1/2}}$, which is the form of a p-series, with $p = 1/2$.
$1/2 < 1$, therefore this series will diverge by the p-series test.

Aside: We can also show it is divergent by the integral test directly, but this is not necessary. Since $k^{-1/2}$ is a positive, decreasing, continuous function on $[1,\infty)$, the series will diverge if the integral $\int_1^{\infty} k^{-1/2}dk$ diverges.
$\lim_{t\to\infty} \int_1^t k^{-1/2}dk = \lim_{t\to\infty} 2t^{1/2} - 2 \to \infty - 2 \to \infty$, which diverges.
Therefore, by the integral test, the series $\sum_{k=1}^{\infty} k^{-1/2}$ also diverges. ■

For these problems, you only need to show the first part from the previous two examples. We just need to recognize p, and whether it is > 1 or ≤ 1 to determine its convergence.

■ **Example 7.26** Determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^5}}$
We notice that $\frac{1}{\sqrt{k^5}} = \frac{1}{k^{5/2}}$, which is a p-series with $p = 5/2$.
Since $5/2 > 1$, this series converges by the p-series test. ■

7.3.4 Translating Indices of Series

You may have noticed that some series thus far have started at $k = 0$ and some have started at $k = 1$. This is actually somewhat arbitrary, aside from the fact that we need to be able to evaluate a_k at each value of k. We can modify the series to start at any specified k value by translating the indices accordingly.

■ **Example 7.27** Translate $\sum_{k=1}^{\infty} \frac{1}{(k+2)^2}$ to start at $k = 3$.

To translate it, we look at the pattern from the first few terms. $\frac{1}{3^2} + \frac{1}{4^2} + \ldots$, and we see that if we start with $k = 3$, the term will shift to $\frac{1}{k^2}$.
So, $\sum_{k=1}^{\infty} \frac{1}{(k+2)^2} = \sum_{k=3}^{\infty} \frac{1}{k^2}$.
The second form is a p=series! We know that it will converge because $p = 2 > 1$. Since the two

series are equivalent, both will converge. ■

7.3.5 Combinations of Series

If a series has multiple terms in its a_k, we can evaluate the terms separately, but if any of them diverge, the entire series will diverge.

■ **Example 7.28** Determine the convergence of $\sum_{k=1}^{\infty} \frac{k^3-1}{k^4}$

We can split the numerator into two terms that we can evaluate: $\sum_{k=1}^{\infty} \left(\frac{k^3}{k^4} - \frac{1}{k^4}\right)$.

Then, we can evaluate each of these separately.

$\sum_{k=1}^{\infty} \frac{k^3}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k}$ is the Harmonic series, which we know is divergent (through integral test or p-series test).

Since this term diverges, it doesn't matter that $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges by the p-series test, because the full series diverges once any of its components diverges.

Thus, $\sum_{k=1}^{\infty} \frac{k^3-1}{k^4}$ diverges. ■

7.3.6 Estimating the Value of an Infinite Series

Earlier, when we discussed the integral test we also learned that the integral does not have the same value as the series, but it behaves similarly. However, we can use the improper integral to bound the remainder of our series.

The remainder is the piece left to calculate after a partial sum: $R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k =$ all remaining terms of the series.

We can use the integral to bound our remainder through $\int_{n+1}^{\infty} f(k)dk \leq R_n \leq \int_{n}^{\infty} f(k)dk$ Generally, we'll use the right side to define an upper bound on our error.

■ **Example 7.29** For the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$, how many terms are needed to ensure the remainder is less than 0.001?

To determine this, we need to solve for n using the upper bound integral.

$\int_{n}^{\infty} \frac{1}{k^3}dk \leq 0.001$

$\lim_{t\to\infty} \int_{n}^{t} \frac{1}{k^3}dk = \lim_{t\to\infty} \frac{-1}{2t^2} + \frac{1}{2n^2} = \frac{1}{2n^2}$

Solve for n: $\frac{1}{2n^2} \leq 0.001$

$1000 \leq 2n^2$

$500 \leq n^2$

$22.36 \leq n$

So we need 23 terms to ensure that the error is less than 0.001.

■

■ **Example 7.30** What is your error bound after 15 terms of $\sum_{k=1}^{\infty} \frac{1}{k^4}$?

The upper bound is found through the integral $\int_{n}^{\infty} \frac{1}{k^4}dk$

$\lim_{t\to\infty} \int_{15}^{t} \frac{1}{k^4}dk = \lim_{t\to\infty} \frac{-1}{3t^3} + \frac{1}{3(15)^3} = 0 + \frac{1}{10,125} \approx 0.0000987654321$

This means our residual after 25 terms is less than 10^{-4} Pretty good! ■

7.3.7 Properties of Convergent Series

1. If $\sum_k a_k$ converges to A, and c is a real number, then $\sum_k ca_k$ converges to cA. (Scalar Multiplication)
2. If $\sum_k a_k$ converges to A, and $\sum_k b_k$ converges to B, then $\sum_k (a_k \pm b_k) = A \pm B$. (Sum of Series = Series' sum)
3. If $\sum_{k=1}^{\infty} a_k$ converges, so will $\sum_{k=m}^{\infty} a_k$ for $m \geq 1$, and vice versa: If the full series converges, then so will any partial sum or subset of the series.

These properties allow us to simplify the series computations, and we've already been using the second one.

■ **Example 7.31** Determine the convergence of the series, and if possible, find its value: $\sum_{k=0}^{\infty} \frac{2-3^k}{6^k}$

We can use property 2 to rewrite this as $\sum_{k=0}^{\infty} \left(\frac{2}{6^k} - \left(\frac{3}{6}\right)^k \right) = \sum_{k=0}^{\infty} 2\left(\frac{1}{6}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

Each of these we can evaluate independently.

$\sum_{k=0}^{\infty} 2\left(\frac{1}{6}\right)^k$ is a geometric series with $a = 2$ and $r = \frac{1}{6}$, this converges because $|r| < 1$, to $\frac{2}{1-1/6} = \frac{12}{5}$

$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ is another geometric series with $a = 1$, and $r = \frac{1}{2}$, this converges also because $|r| < 1$, to $\frac{1}{1-1/2} = 2$

Then, $\sum_{k=0}^{\infty} \frac{2-3^k}{6^k} = \frac{12}{5} - 2 = \frac{2}{5}$. The series converges! ■

7.4 The Comparison Tests

Now we have some known forms for series that converge, and so we can use series that we know the convergence of, to compare with unknown series. These form the Comparison tests.
The first comparison test is called The Comparison Test. We directly compare two series through an inequality, and use the convergence of the known series to define the behavior of the unknown series.

Given two positive series $\sum_k a_k$ and $\sum_k b_k$:
If $0 < a_k \leq b_k$ for all k, and $\sum_k b_k$ converges, then so will $\sum_k a_k$ (it's less than a convergence series, so it must also converge).
If $0 < b_k \leq a_k$ for all k, and $\sum_k b_k$ diverges, then so will $\sum_k a_k$ (it's greater than a divergent series, so it must also diverge).

■ **Example 7.32** Use the comparison test to determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{3k^3+2}$

We suspect that this will converge because it looks similar to a p-series with $p = 3$, but we need to show that it converges.
So, we seek a series we can evaluate to compare with.
Specifically, we use $\sum_{k=1}^{\infty} \frac{1}{3k^3} = \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{k^3}$ which using property 1 above, implies that the value this series converges to will just be multiplied by $\frac{1}{3}$. This is a p-series with $p = 3$ and $3 > 1$, so we

know it converges by the p-series test.
We just need to show that $\sum_{k=1}^{\infty} \frac{1}{3k^3+2} \leq \sum_{k=1}^{\infty} \frac{1}{3k^3}$.
To do this we look at the terms: $\frac{1}{3k^3+2} \leq \frac{1}{3k^3}$? Well, if we simplify, this says $3k^3 \leq 3k^3+2$, which is true for all values of k in the series.
Therefore, since $\sum_{k=1}^{\infty} \frac{1}{3k^3}$ converges, and $\sum_{k=1}^{\infty} \frac{1}{3k^3+2} \leq \sum_{k=1}^{\infty} \frac{1}{3k^3}$, then $\sum_{k=1}^{\infty} \frac{1}{3k^3+2}$ must also converge by the Comparison Test. ■

■ **Example 7.33** Use the comparison test to determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{4k^2+1}}$
We again suspect that this will converge because it looks like a p-series with $p=2$ (if we were to ignore the added constant in the square root).
So, we seek a similar p-series to compare it with.
Specifically, we will use $\sum_{k=1}^{\infty} \frac{1}{2k^2}$, which converges by the p-series test with $p=2>1$.
We then need to show that $\frac{1}{k\sqrt{4k^2+1}} \leq \frac{1}{2k^2}$ for all k.
$2k^2 \leq k\sqrt{4k^2+1}$?
$2k \leq \sqrt{4k^2+1}$?
$4k^2 \leq 4k^2+1$? Yes, the terms of our convergent series are all greater than the terms of our unknown series.
Therefore, since $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ converges, and $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{4k^2+1}} \leq \sum_{k=1}^{\infty} \frac{1}{2k^2}$, then $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{4k^2+1}}$ must also converge by the Comparison Test. ■

■ **Example 7.34** Use the comparison test to determine the convergence of $\sum_{k=3}^{\infty} \frac{\ln(k)}{k}$
We suspect that this will diverge because it is larger than the Harmonic series, which also diverges.
We can compare these directly to show that the series diverges.
$\frac{\ln(k)}{k} \geq \frac{1}{k}$?
$\ln(k) \geq 1$? For all k in the series? (note: that starts at $k=3$) Yes. The terms of our unknown series are all greater than the terms of our known, divergent series.
Therefore, since $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges, and $\sum_{k=3}^{\infty} \frac{\ln(k)}{k} \geq \sum_{k=3}^{\infty} \frac{1}{k}$, then $\sum_{k=3}^{\infty} \frac{\ln(k)}{k}$ must also diverge by the Comparison Test. ■

7.4.1 The Limit Comparison Test

The Comparison test used a direct, algebraic comparison. The Limit Comparison test uses a limit of the ratio of two series' terms to determine convergence.

Given two positive series $\sum_k a_k$ (given / unknown convergence) and $\sum_k b_k$ (known convergence), we evaluate the limit $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, and based on the known series and the value of L, we can determine the convergence of the unknown series.

Cases:

1. If L is a positive, finite number, then both series have the same convergence. (If $\sum_k b_k$ converges, then so does $\sum_k a_k$. If $\sum_k b_k$ diverges, then so does $\sum_k a_k$)
2. If $L=0$ and $\sum_k b_k$ converges, then so will $\sum_k a_k$
3. If $L\to\infty$ and $\sum_k b_k$ diverges, then so will $\sum_k a_k$

■ **Example 7.35** Determine the convergence of $\sum_{k=1}^{\infty} \frac{3k^3-2k^2}{k^5-k^3}$ using the limit comparison test

The leading terms look like $\frac{3k^3}{k^5} = \frac{3}{k^2}$, so a natural choice to compare with is $\sum_{k=1}^{\infty} \frac{3}{k^2}$, which we know converges by the p-series test

$$\lim_{k\to\infty} \frac{\frac{3k^3-2k^2}{k^5-k^3}}{\frac{3}{k^2}} = \lim_{k\to\infty} \frac{3k^5-2k^4}{3k^5-3k^3} \to \frac{\infty}{\infty}$$

We can apply L'Hopital's rule to evaluate this, $\lim_{k\to\infty} \frac{15k^4-8k^3}{15k^4-9k^2} \to \frac{\infty}{\infty}$

We apply L'Hopital's rule again, $\lim_{k\to\infty} \frac{60k^3-24k^2}{60k^3-18k} \to \frac{\infty}{\infty}$

And, we apply L'Hopital's rule again, $\lim_{k\to\infty} \frac{180k^2-48k}{180k^2-18} \to \frac{\infty}{\infty}$

And again, $\lim_{k\to\infty} \frac{360k-48}{360k}$, which simplifies to $\lim_{k\to\infty} 1 - \frac{48}{360k} = 1$.

Since 1 is a defined constant, we know that the two series have the same convergence. Since $\sum_{k=1}^{\infty} \frac{3}{k^2}$ converges, $\sum_{k=1}^{\infty} \frac{3k^3-2k^2}{k^5-k^3}$ also converges by the Limit Comparison Test. ■

■ **Example 7.36** Determine the convergence of $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ using the limit comparison test

We seek a known series, so choosing a p-series is a good start. We will start by comparing with $\sum_{k=1}^{\infty} \frac{1}{k}$, which we know diverges.

$$\lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\ln(k)}{k} \to \frac{\infty}{\infty}$$

We can then apply L'Hopital's rule, $\lim_{k\to\infty} \frac{\frac{1}{k}}{1} \to 0$

However, our known series diverges, so this result is inconclusive.

We will then try $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which we know converges by the p-series test.

$$\lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \ln(k) \to \infty$$

However, since our known series converges, this result is also inconclusive.

So, let's try a p-series between these two. $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$, which we know converges by the p-series test.

$$\lim_{k\to\infty} \frac{\frac{\ln(k)}{k^2}}{\frac{1}{k^{3/2}}} = \lim_{k\to\infty} \frac{\ln(k)}{k^{1/2}} \to \frac{\infty}{\infty}$$

We can then apply L'Hopital's rule, $\lim_{k\to\infty} \frac{\frac{1}{k}}{1/2k^{-1/2}} = \lim_{k\to\infty} \frac{2}{k^{1/2}} \to 0$

Finally! Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges by the p-series test, $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$ also converges by the Limit Comparison Test. ■

7.5 Alternating Series

We discussed alternating sequences in the introduction material, and we talked about a sequence converging by oscillation. However, we can also have an alternating series. All of our previous tests were specifically written for non-alternating series (series with all positive terms). The terms of an alternating series oscillate, and so when we add the terms together there are actually less requirements for convergence of an alternating series.

We consider the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$. Note: we are keeping the a_k as a positive term, and treating the oscillation separately through the $(-1)^{k+1}$.

As you recall from earlier, the Harmonic series diverges. However, the Alternating Harmonic series actually does converge, $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$. Since the terms of this series oscillate about 0, the series converges much quicker: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...$

7.5.1 The Alternating Series Test

If 1) $a_{k+1} \leq a_k$ for all $k \geq N$
and 2) $\lim_{k\to\infty} a_k = 0$
Then, the alternating series converges!
(Much easier to show than the positive series' tests)

■ **Example 7.37** Determine if $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{8^k}$ converges
Since we see the $(-1)^{k+1}$ form, we know this is an alternating series, so we check for the two properties to determine convergence.
1) Is $\frac{k+1}{8^{k+1}} \leq \frac{k}{8^k}$?
Simplify algebraically: $\frac{k+1}{k} \leq 8$?
Check: $\frac{2}{1} = 2 < 8$, $\frac{3}{2} = 1.5 < 8$, $\frac{4}{3} = 1.3333 < 8$, clearly, as $k \to \infty$, $\frac{k+1}{k} \to 1$. So, it is always less than 8 - great!
2) Evaluate $\lim_{k\to\infty}\frac{k}{8^k} \to \frac{\infty}{\infty}$
We apply L'Hopital's rule, $\lim_{k\to\infty}\frac{1}{8^k\ln(8)} \to 0$ Yes!

Both properties of the Alternating Series Test are satisfied, therefore $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{8^k}$ converges by the A. S. T. ■

■ **Example 7.38** Determine if $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\ln(k)}{k}$ converges
Since this is an alternating series, we apply the alternating seriest test.
1) Is $\frac{\ln(k+1)}{k+1} \leq \frac{\ln(k)}{k}$?
Simplify: $\frac{\ln(k+1)}{\ln(k)} \leq \frac{k+1}{k}$?
This is clearly not true for $k = 1$, but is it true after a specified value? All we need is a value N where this is true for all values after it.
Check: $\frac{\ln(3)}{\ln(2)} \leq \frac{3}{2}$? No.

$\dfrac{\ln(4)}{\ln(3)} \leq \dfrac{4}{3}$? Yes!, and because $\ln(k)$ grows slower than k, this will be true for all $k \geq 3$.

2) Evaluate $\lim_{k\to\infty} \dfrac{\ln(k)}{k} \to \dfrac{\infty}{\infty}$

We apply L'Hopital's rule, $\lim_{k\to\infty} \dfrac{\frac{1}{k}}{1} = 0$ Yes!

Both properties of the Alternating Series Test are satisfied, therefore $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}\ln(k)}{k}$ converges by the A. S. T. ■

7.5.2 Remainder of an Alternating Series

Since our alternating series terms tend to oscillate about 0 as they converge, the remainder term is no larger than the next term in the series.

$$R_n = |S - S_n| \leq a_{n+1}$$

■ **Example 7.39** Approximate $S = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^6}$ to 4 decimal places (10^{-4})
We can use our bound on the remainder term to find a k value that will ensure this accuracy.
$R_n \leq a_{n+1} = \dfrac{1}{(k+1)^6} \leq \dfrac{1}{10^4}$
So, $(k+1)^6 \geq 10^4$
$k+1 \geq \sqrt[6]{10^4} \approx 4.64$
So, $k \geq 3.64$, which means if we use 4 terms (always round up!), we can ensure that our remainder is less than 10^{-4}, so we will be accurate to 4 decimal places.
To approximate that value, we just add up the first 4 terms: $1 - \dfrac{1}{64} + \dfrac{1}{729} - \dfrac{1}{4096} \approx 0.9855$ (to the first 4 decimal places) ■

■ **Example 7.40** How accurate is the approximation of $S = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{2^k}$ after 10 terms?
$R_n \leq a_{n+1} = \dfrac{1}{2^11} = \dfrac{1}{2048} \approx 4.9 \times 10^{-4}$.

How many terms do you need for accuracy of 10^{-3}?
$R_n \leq a_{n+1} = \dfrac{1}{2^{n+1}} \leq \dfrac{1}{10^3}$
Simplifying, we have $2^{n+1} \geq 1000$
We can apply a natural logarithm to both sides to simplify further, $(n+1)\ln(2) \geq \ln(1000)$
$n+1 \geq \dfrac{\ln(1000)}{\ln(2)} \approx 9.966$
So, $n \geq 8.966$, or $n = 9$ will ensure accuracy of 10^{-3}. ■

7.5.3 Absolute and Conditional Convergence

Now that we are working with alternating series as well as positive series, we need to distinguish the convergence. Recall that $\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges, but $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}$ converges. This is an example of a conditionally convergent series. This is conditionally convergent because it does converge if it is an alternating series, but not if it is non-alternating.

We use this concept to define that a series is absolutely convergent if $\sum_k |a_k|$ converges.

Example 7.41 Determine if the following series are absolutely convergent, or conditionally convergent:

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ln(k)}{k}$ (we discussed this one in an earlier example), this series converges as it is, by the A. S. T. However, the positive series $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ diverges. So, the series is conditionally convergent.
2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$, first we check if the alternating series converges:
   1) Is $\frac{1}{(k+1)^2} \leq \frac{1}{k^2}$ for all $k \geq N$? Yes.
   2) Evaluate $\lim_{k\to\infty} \frac{1}{k^2} = 0$ Great!
   The alternating series converges by the A. S. T.
   Then, we determine the convergence of the positive series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges by the p-series test, with $p = 2 > 1$. Since the positive series also converges, this is an absolutely convergent series.
3. $\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k}$, first we check if the alternating series converges:
   1) Is $e^{-(k+1)} \leq e^{-k}$ for all $k \geq N$? Yes.
   2) Evaluate $\lim_{k\to\infty} e^{-k} = 0$ Great!
   The alternating series converges by the A. S. T.
   Then, we determine the convergence of the positive series $\sum_{k=1}^{\infty} e^{-k}$.
   We see that e^{-k} is a positive, decreasing, continuous function, so we can apply the Integral test.
   $\int_1^{\infty} e^{-k}dk = \lim_{t\to\infty}\int_1^t e^{-k}dk = \lim_{k\to\infty} -e^{-t} + e^{-1} = e^{-1}$ The limit exists, so both the integral and the series converge!
   Since the positive series also converges, this is an absolutely convergent series.

■

7.6 The Ratio and Root Tests

The Ratio and Root tests are additional tests to determine the convergence of series, these are each particularly good for specific types of series.

7.6.1 The Ratio Test

The Ratio test uses a comparison of the terms from a positive, infinite series, to determine its convergence. Specifically, we look at the $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$, so this test is particularly good for exponential and factorial functions because terms will cancel and simplify the limit calculation.

To determine the convergence of a given series $\sum_k a_k$, we define $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.
If $0 \leq r < 1$, the series converges.
If $r = 1$, the test is inconclusive.
If $r > 1$, the series diverges.

Example 7.42 Apply the Ratio test to determine the convergence of $\sum_{k=1}^{\infty} 2^k$
$\lim_{k\to\infty} \frac{2^{k+1}}{2^k} = \lim_{k\to\infty} 2 = 2$.
Since $r = 2 > 1$, this series diverges by the Ratio test. ■

■ **Example 7.43** Apply the Ratio test to determine the convergence of $\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k$

$\lim_{k\to\infty}\frac{\left(\frac{1}{3}\right)^{k+1}}{\left(\frac{1}{3}\right)^k} = \lim_{k\to\infty}\frac{1}{3} = \frac{1}{3} < 1$, so this series converges by the Ratio test. ■

■ **Example 7.44** Apply the Ratio test to determine the convergence of $\sum_{k=1}^{\infty}\frac{2^k}{k!}$

$\lim_{k\to\infty}\frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}} = \frac{2^{k+1}k!}{2^k(k+1)!}$

Recall: factorial function. $n! = n(n-1)(n-2)(n-3)...(3)(2)(1)$. So, $3! = (3)(2)(1) = 6$, for example.

So, this means that when we look at the simplification, we can use the fact that $(n+1)! = n!(n+1)$ to simplify to $\lim_{k\to\infty}\frac{2}{k+1} = 0 < 1$, so this series converges by the Ratio test. ■

■ **Example 7.45** Apply the Ratio test to determine the convergence of $\sum_{k=1}^{\infty}\frac{k}{2k^2+1}$

$\lim_{k\to\infty}\frac{\frac{k+1}{2(k+1)^2+1}}{\frac{k}{2k^2+1}} = \lim_{k\to\infty}\frac{(k+1)(2k^2+1)}{k(2(k+1)^2+1)} = \lim_{k\to\infty}\frac{2k^3+2k^2+k+1}{2k^3+4k^2+3k} \to \frac{\infty}{\infty}$

We can then apply L'Hopital's rule, until we reach a conclusion.

$\lim_{k\to\infty}\frac{6k^2+4k+1}{6k^2+8k+3} \to \frac{\infty}{\infty}$

$\lim_{k\to\infty}\frac{12k+4}{12k+8} \to \frac{\infty}{\infty}$

$\lim_{k\to\infty}\frac{12}{12} = 1$. This is the inconclusive case, we cannot determine the convergence of $\sum_{k=1}^{\infty}\frac{k}{2k^2+1}$ using the Ratio test.

Luckily, we have other tests. This is a positive, decreasing, continuous function on $[1,\infty)$, so we can apply the integral test.

$\int_1^{\infty}\frac{k}{2k^2+1}dk = \lim_{t\to\infty}\int_1^t\frac{k}{2k^2+1}dk$

We can integrate this by using a u-substitution: $u = 2x^2+1$, so $du = 4xdx$, updating our integral: $\lim_{t\to\infty}\int_3^{2t^2+1}\frac{1}{4}\frac{1}{u}du = \lim_{t\to\infty}\frac{1}{4}\ln|u||_3^{2t^2+1} = \lim_{t\to\infty}\frac{1}{4}\ln|2t^2+1| - \frac{1}{4}\ln|3| \to \infty$, so the limit and integral diverge, which tells us that the series $\sum_{k=1}^{\infty}\frac{k}{2k^2+1}$ also diverges by the integral test. ■

7.6.2 The Root Test

This is the last series test! Hooray! It's also somewhat of a last resort test, unless you see something of the form $(bob)^k$ inside of your series. Then, it is a great choice!

For a positive series, $\sum_k a_k$, we define $\rho = \lim_{k\to\infty}\sqrt[k]{a_k}$.

Similar to the Ratio test, we define convergence of our series through the value of ρ:

If $0 \le \rho < 1$, the series is convergent.

If $\rho = 1$, the test is inconclusive.

If $\rho > 1$, the series is divergent.

■ **Example 7.46** Apply the Root test to determine the convergence of $\sum_{k=1}^{\infty} \left(\frac{3k+1}{4k-2}\right)^k$

Evaluate $\lim_{k\to\infty} \sqrt[k]{\left(\frac{3k+1}{4k-2}\right)^k} = \lim_{k\to\infty} \frac{3k+1}{4k-2} \to \frac{\infty}{\infty}$

We can apply L'Hopital's rule, $\lim_{k\to\infty} \frac{3}{4} = \frac{3}{4} < 1$, so the series is convergenty by the Root test. ■

■ **Example 7.47** Apply the Root test to determine the convergence of $\sum_{k=1}^{\infty} \frac{3^k}{k^{15}}$

Evaluate $\lim_{k\to\infty} \sqrt[k]{\frac{3^k}{k^{15}}} = \lim_{k\to\infty} \frac{3}{k^{15/k}}$

We can rewrite $k^{15/k}$ as $e^{15\ln(k)/k}$ which approaches $e^{\infty/\infty}$.

If we evaluate that ratio specifically, we have $\lim_{k\to\infty} \frac{15\ln(k)}{k} \to$ L'Hopital's rule $\to \lim_{k\to\infty} \frac{\frac{15}{k}}{1} \to 0$, and $e^0 = 1$.

Plugging this back into our limit, we find that $\lim_{k\to\infty} \frac{3}{k^{15/k}} = \lim_{k\to\infty} \frac{3}{1} = 3 > 1$, so this series diverges by the Root test. ■

7.7 Choosing a Convergence Test

This section is mostly about strategy, the text has a table comparing the series methods and types.

Look to identify what type of series you are working with - some tests are best for certain types of series. Such as: Geometric series, telescopint series (partial fraction decomposition), p-series, exponentials, and/or factorials.

Strategies: Tests in order of application/use.

1. The Divergence test: this is a quick test for divergence only.
   If $\lim_{k\to\infty} a_k \neq 0$ the series diverges, and you are done.
   If $\lim_{k\to\infty} a_k = 0$, then the test is INCONCLUSIVE. You need to try another test.
2. Special series:
   (a) Geometric Series: these are of the form $\sum_{k=0}^{\infty} ar^k$, or $\sum_{k=1}^{\infty} ar^{k-1}$
   The geometric series converges if $|r| < 1$
   The geometric series diverges if $|r| \geq 1$
   (b) P-series: these are of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$
   The p-series converges if $p > 1$
   The p-series diverges if $p \leq 1$
   (c) Telescoping series: these are of the form $\sum_{k=1}^{\infty} \left(\frac{A}{k+a} - \frac{B}{k+b}\right)$, or $\sum_{k=1}^{\infty} \frac{C}{(k+a)(k+b)}$
   (which requires partial fraction decomposition to form the telescoping action).
   In these series, the middle terms cancel out due to the positive and negative terms that are of the same form.
   (d) Alternating series: these are of the form $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$
   In these series, we only need to show two properties:
   1) $a_{k+1} \leq a_k$ for all $k \geq N$
   2) $\lim_{k\to\infty} a_k = 0$

3. The Integral test: only applicable if the function is positive, decreasing, and continuous on the interval.
   When the requirements are met, we evaluate the improper integral on the same interval, and if that integral converges, so does the series.
4. The Ratio test: particularly useful for exponential and factorial functions (especially if you have factorials! You want to use the ratio test)
   $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.
   If $0 \le r < 1$, the series converges.
   If $r = 1$, the test is inconclusive.
   If $r > 1$, the series diverges.
5. The Comparison test: A direct algebraic comparison of two series, one known that you choose because it is similar to the other, and the other you are given to determine its convergence.
   If the given series is less than the known series, and the known series converges, then so does the given series.
   If the given series is greater than the known series, and the known series diverges, then so does the given series.
6. The Limit Comparison test: A ratio of two series terms, one known that you choose because it is similar to the other, and the other you are given to determine its convergence.
   $\lim_{k\to\infty} \frac{a_k}{b_k} = L$
   If L is a positive, finite number, then both series have the same convergence. (If $\sum_k b_k$ converges, then so does $\sum_k a_k$. If $\sum_k b_k$ diverges, then so does $\sum_k a_k$)
   If $L = 0$ and $\sum_k b_k$ converges, then so will $\sum_k a_k$
   If $L \to \infty$ and $\sum_k b_k$ diverges, then so will $\sum_k a_k$
7. The Root test: best for cases where we have $(bob)^k$ where bob is not a constant. Can also be used if you find yourself stuck, and need to try another technique.
   $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$
   If $0 \le \rho < 1$, the series is convergent.
   If $\rho = 1$, the test is inconclusive.
   If $\rho > 1$, the series is divergent.

The best strategies for YOU will come from your own practice. This is a skill, and not something you can memorize to complete. You must practice, A LOT, in order to solidify these concepts enough to do well on an exam. So, go solve some problems!!!

8. Approximating Functions

This chapter contains the building blocks for approximation of functions, which are used extensively in discrete approximations.

8.1 Approximating Functions with Polynomials

The real world is messy, and the functions we often end up working with can be so complicated that we prefer to use approximations to get reasonable results. There is also the case where we are modeling data, and we cannot define a function for the observation, so we approximate a function that matches our data in order to make predictions or approximations to values outside our observation.

There is also the fact that there are still several functions we cannot integrate using the techniques available to us in this course. We can always fall back on numerical integration, but that is only so accurate for approximating the integral of a function. We can also approximate that function in the region of integration through a polynomial approximation, and integrate the polynomial to approximate the integral of the actual function. This is why we introduce the concept of polynomial approximations to general functions.

By approximating the function with a polynomial, we have a function we can integrate exactly. Polynomials are a nice choice because they are easy to integrate, and they are continuous everywhere.

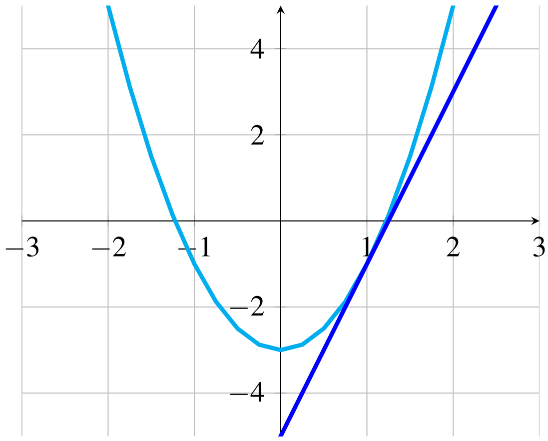
In this chapter, we will delve deeper into our series notation, and use it to represent these approximations. One of the biggest takeaways from this chapter should be Power series, which is a series representation of a function built through powers of x. $\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots$. This power series form is for functions we call 'centered about $x = 0$", more generally we use $\sum_{k=0}^{\infty} c_k (x-a)^k$, which forms a series centered at $x = a$.

8.1.1 Linear Approximation

The first order approximation you have already done in Calculus I: Linear approximation. We used the tangent line in order to approximate a function based on its slope and its value at a given value of x.

$$y = f(a) + f'(a)(x-a)$$

■ **Example 8.1** Determine the linear approximation to $f(x) = 2x^2 - 3$, centered at $x = 1$ $(a = 1)$
$f(1) = 2-3 = -1$, and $f'(x) = 4x$, so $f'(1) = 4$
We then form the linear approximation $L(x) = -1 + 4(x-1) = 4x - 5$ ■



The linear approximation forms a line that passes through the same point as $f(x)$, with the same slope.

8.1.2 Quadratic Approximation

To build a quadratic approximation, we now need to match the second derivative of $f(x)$ with our approximation, $Q(x) = f(a) + f'(a)(x-a) + c_2(x-a)^2$.

$$Q''(a) = 2c_2 = f''(a), \text{ so } c_2 = \frac{f''(a)}{2}$$

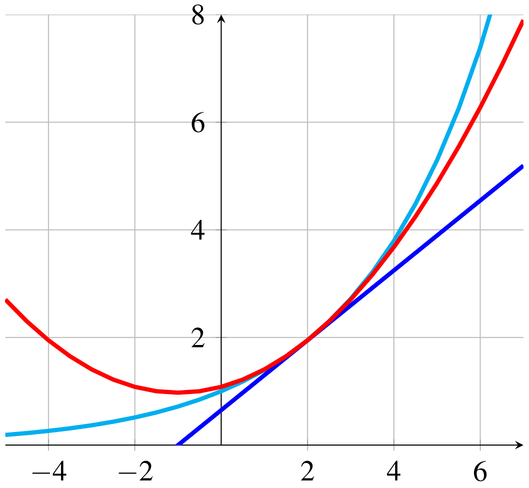
This constructs our quadratic approximation to a given function, $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

■ **Example 8.2** Approximate $f(x) = e^{x/3}$ using a linear and quadratic approximation centered at $x = 2$.
$L(x) = f(2) + f'(2)(x-2) = e^{2/3} + \frac{1}{3}e^{2/3}(x-2)$
$Q(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 = e^{2/3} + \frac{1}{3}e^{2/3}(x-2) + \frac{1}{18}e^{2/3}(x-2)^2$

■

8.1.3 Taylor Polynomials

Taylor polynomials are tformed in the same manner we constructed our linear and quadrativ approximations, but extended to the general form:



The cyan curve is the original function $e^{x/3}$, the blue curve is the linear approximation, and the red curve is the quadratic approximation. From the graph, you can see that the red curve makes a better approximation to the function for a wider interval.

$P_n(x) = \sum_{k=0}^{n} c_k(x-a)^k$, where each $c_k = \dfrac{f^{(k)}(a)}{k!}$ (the superscript in parentheses is to denote the derivative taken)

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We refer to this as the “nth order Taylor Polynomial"

■ **Example 8.3** Find the 4th order Taylor Polynomial of $f(x) = \ln(x)$ centered about $x = 1$ $(a = 1)$
To construct the polynomial, we first compute the associated derivatives needed for the approximation.
$f'(x) = \dfrac{1}{x}$, $f''(x) = \dfrac{-1}{x^2}$, $f'''(x) = \dfrac{2}{x^3}$, $f^{(4)}(x) = \dfrac{-6}{x^4}$
Then, we evaluate them at our a to get our coefficients
$f(1) = \ln(1) = 0$, $f'(1) = \dfrac{1}{1} = 1$, $f''(1) = \dfrac{-1}{1^2} = -1$, $f'''(1) = \dfrac{2}{1^3} = 2$, $f^{(4)}(1) = \dfrac{-6}{1^4} = -6$
Substituting these into the general form yields our polynomial:
$P_4(x) = 0 + 1(x-1) + \dfrac{-1}{2}(x-1)^2 + \dfrac{2}{3!}(x-1)^3 + \dfrac{-6}{4!}(x-1)^4$
This simplifies to: $P_4(x) = (x-1) - \dfrac{1}{2}(x-1)^2 + \dfrac{1}{3}(x-1)^3 - \dfrac{1}{4}(x-1)^4$ (but there is no need to simplify further. ■

Do not expand these approximations, it will take a lot of work, and only make them harder to evaluate when approximating function values.

We can use the approximation from our example to then approximate the value of $\ln(2)$.

$$P_4(2) = (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} \approx 0.5833$$

We can compare this to the actual value of $\ln(2) \approx 0.693147$, which we can see is not a great approximation. Why is it not a great approximation?

Well, first, we are approximating a value that isn't very close to the original point. If we use the approximation to estimate the value of $\ln(1.1)$ we will have significantly better results. Additionally, we can improve on our approximation through adding more terms to the polynomial approximant. More terms = better approximation to the function, so long as the Taylor Polynomial is actually converging on the function as $n \to \infty$.

8.1.4 Remainder of a Taylor Polynomial

The remainder of the Taylor Polynomial is treated much the same as the remainder in our series approximation through partial sums.

$R_n(x) = f(x) - P_n(x)$, and this remainder is bounded by the next term in the Taylor polynomial.

Theorem 8.1.1 Taylors Theorem: for a function $f(x)$ with continuous derivatives, and $P_n(x)$ centered at $x = a$

$R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where c is a value on the interval $[a,x]$

For the same example, estimate the remainder in our approximation of $\ln(2)$.

Since we do not know the exact value of c in $[a,x]$ that makes the equality, we instead bound our error by maximizing the derivative on our interval.

Find $M = \max_{c\in[a,x]}\left(f^{(n+1)}(c)\right)$ (the value c which maximizes the $n+1$th derivative on $[a,x]$

$f^{(5)}(x) = \dfrac{24}{x^5}$, since we approximated $\ln(2)$, we find the value that maximizes this function on $[1,2]$.

Since it is largest when $x = 1$, we will use that value to estimate our remainder, $M = \dfrac{24}{1^5} = 24$.

$$|R_n(x)| \leq M\frac{(x-a)^n}{n!}$$

For our problem, $|R_4(2)| \leq 24\dfrac{(x-1)^5}{5!} = \dfrac{1}{5} = 0.2$ error at $x = 2$, in our approximation.

This is large, but also completely consistent with our actual error at that point.

Actual error was $|0.693147 - 0.5833| = 0.1098 < 0.2$

Practice! You will use these a lot in higher-level math courses! Taylor Polynomials are the foundation for a lot of formulas used in lower level math courses, and STEM courses.

8.2 Power Series

Taylor polynomials are a specific form of power series. Power series are polynomial series representation of a function. They are of the form: $\sum_{k=0}^{\infty} cx^k$

Recall: Geometric series $\sum_{k=0}^{\infty} cr^k = \dfrac{c}{1-r}$ if $|r| < 1$

This generalizes for the convergence of our power series functions, $\sum_{k=0}^{\infty} cx^k = \dfrac{c}{1-x}$ for $|x| < 1$

This allows us to define the values of x for which the power series converges. In the case where our series is exactly of the form above, it is only convergent for values of $|x| < 1$, so it has an interval of convergence $(-1,1)$. We define the radius of convergence from the interval of values $(-1,1)$, the width of the interval is $(1-(-1)) = 2$, and so the radius is 1.

Note: this series form does not have an $(x-a)$ term, so it is centered at $x = 0$.

We represent the more general form of our power series through $\sum_{k=0}^{\infty} c_k(x-a)^k$, as mentioned in the previous section.

This causes the interval to shift so that it is centered at $x = a$, but the radius is still going to be defined by the width of the interval. Generally $I = (a-R, a+R)$ where I is the interval of convergence, and R is the radius of convergence.

In order to determine the convergence for these power series, we will use the Ratio test from earlier: $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right|$. The result of this limit will determine the convergence of our series.

There are 3 primary cases for convergence of a power series:

1. When the limit diverges, the series converges only when $x = a$, so it is divergent generally. In this case, there is no interval of convergence because there is only one point where it converges. So, $R = 0$.
2. When the limit converges to 0, the series converges for all x values, so the interval of convergence is $(-\infty, \infty)$, and $R = \infty$
3. The limit converges to a function of x, so that $|f(x)| < 1$. Then, the series converges for a fixed interval of values, so the interval of convergence is $(a-R, a+R)$, where $R =$ a defined constant value.
   This case is special because we occasionally see one or both endpoints included in the interval of convergence (I can equal $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$). So, in order to ensure we have the correct interval, we need to check the endpoints individually to determine their convergence.

■ **Example 8.4** Find the radius and interval of convergence for the following series: $\sum_{k=0}^{\infty} k!x^k$
We apply the ratio test to determine the radius of convergence
$\lim_{k\to\infty} \left|\dfrac{(k+1)!x^{k+1}}{k!x^k}\right| = \lim_{k\to\infty} |(k+1)x| \to \infty$ this is an example of the case when our ratio test diverges, except for the case when $x = 0$, so $R = 0$, and there is no interval of convergence, just the point at $x = 0$ ■

■ **Example 8.5** Find the radius and interval of convegence for the following series: $\sum_{k=0}^{\infty} \dfrac{x^k}{k!}$
We apply the ratio test to determine the radius of convergence

$\lim_{k\to\infty} \left|\dfrac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}\right| = \lim_{k\to\infty} \left|\dfrac{x}{k+1}\right| = 0$, so this limit converges regardless of the x value!

Thus, $R = \infty$, and our interval of convergence is $(-\infty, \infty)$ ■

■ **Example 8.6** Find the radius and interval of convergence for the power series $\sum_{k=1}^{\infty} (-1)^k kx^k$
Recall: Translating series. This given series starts at $k = 1$, instead of $k = 0$. Before we evaluate it,

we need to translate it to start at $k=0$.
$\sum_{k=1}^{\infty}(-1)^k kx^k = \sum_{k=0}^{\infty}(-1)^{k+1}(k+1)x^{k+1}$
We apply the ratio test to determine its convergence: $a_k=(k+1)x^{k+1}$, $a_{k+1}=(k+2)x^{k+2}$
$\lim_{k\to\infty}\left|\dfrac{(k+2)x^{k+2}}{(k+1)x^{k+1}}\right| = \lim_{k\to\infty}\left|\dfrac{(k+2)x}{(k+1)}\right| = |x|<1$ for convergence by the Ratio test.
So, this power series is convergent for $|x|<1$ and $R=1$. However, this is case 3, so we need to check the endpoints to be sure we have the correct interval.
$x=-1$: $\sum_{k=0}^{\infty}(-1)^{k+1}(k+1)(-1)^{k+1} = \sum_{k=1}^{\infty} k+1$
We can apply the divergence test to show that this series diverges: $\lim_{k\to\infty} k+1 \to \infty \neq 0$, so the series diverges by the divergence test.
$x=1$: $\sum_{k=0}^{\infty}(-1)^{k+1}(k+1)(1)^{k+1} = \sum_{k=0}^{\infty}(-1)^{k+1}(k+1)$
This is an alternating series, so we can apply the alternating series test to determine its convergence.
1) Is $k+2<k+1$ for all $k\geq N$? No. The second property is also not satisfied, but just this first one is enough to say that the alternating series diverges by the A. S. T.
Therefore, our interval of convergence is $(-1,1)$ ■

■ **Example 8.7** Determine the radius and interval of convergence of the power series $\sum_{k=0}^{\infty}\dfrac{(x-2)^k}{k}$
We apply the ratio test to determine its convergence:
$\lim_{k\to\infty}\left|\dfrac{\frac{(x-2)^{k+1}}{k+1}}{\frac{(x-2)^k}{k}}\right| = \lim_{k\to\infty}\left|\dfrac{(x-2)k}{k+1}\right| = |x-2|<1$ for convergence.
So, our interval is defined by $-1<x-2<1$, so $1<x<3$, and our radius of convergence is $R=1$.
Check endpoints!
$x=1$: $\sum_{k=0}^{\infty}\dfrac{(-1)^k}{k}$. This is the Alternating Harmonic Series, which we determined converges by the Alternating Series Test in that section. So, this endpoint is included!
$x=3$: $\sum_{k=0}^{\infty}\dfrac{1^k}{k}$. This is the Harmonic Series, which we determined diverges by the integral test in that section. So, this endpoint is excluded.
Our interval of convergence is then $I=[1,3)$ ■

8.2.1 Convergence of Taylor Polynomials

We can now combine the Taylor Polynomial approximations with our analysis of the convergence of our power series, and determine the convergence of our Taylor Polynomial to a given function.

■ **Example 8.8**
1. Approximate $f(x)=\sin(x)$ with a 4th order Taylor Polynomial, centered about $x=\dfrac{\pi}{2}$ $(a=\dfrac{\pi}{2})$.
2. Use $P_4(x)$ to approximate $f\left(\dfrac{\pi}{4}\right)$, and evaluate its error in comparison to the bound on its remainder.
3. Using the form of the 4th order Taylor Polynomial, determine a general form for the power series representation of the function, and determine its radius and interval of convergence.

1. In order to build $P_4(x)$, we need to evaluate the function and its first 4 derivatives at the given x value.
$f'(x)=\cos(x)$, $f''(x)=-\sin(x)$, $f'''(x)=-\cos(x)$, $f^{(4)}(x)=\sin(x)$
$f\left(\frac{\pi}{2}\right)=1$, $f'\left(\frac{\pi}{2}\right)=0$, $f''\left(\frac{\pi}{2}\right)=-1$, $f'''\left(\frac{\pi}{2}\right)=0$, and $f^{(4)}\left(\frac{\pi}{2}\right)=1$
So, $P_4(x)=1+0\left(x-\frac{\pi}{2}\right)-\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2-0\left(x-\frac{\pi}{2}\right)^3+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4$
Which simplifies nicely to $P_4(x)=1-\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{24}\left(x-\frac{\pi}{2}\right)^4$

2. $P_4\left(\frac{\pi}{4}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(\frac{\pi}{4} - \frac{\pi}{2}\right)^4 \approx 0.70743$
   The actual value of $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \approx 0.707107$
   So, the error is $|0.707107 - 0.70743| = 0.000323$ Great!
   The remainder bound is determined through $f^{(5)}(x) = \cos(x)$ at its maximum on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, which happens at $x = \frac{\pi}{4}$, where $f^{(5)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = M$
   So, $R_4(x) \leq M \frac{\left|x - \frac{\pi}{2}\right|^5}{5!}$, and $R_4\left(\frac{\pi}{4}\right) \leq \frac{\sqrt{2}}{2} \frac{\left|\frac{\pi}{4} - \frac{\pi}{2}\right|^5}{5!} \approx 0.001761$
   This is consistent because $0.001761 > 0.000323$ so our bound is greater than the actual error, as it should be!
3. Note the patterns: We only have even powers of $\left(x - \frac{\pi}{2}\right)$, the sign of each term oscillates, and the coefficients are just 1 divided by the associated factorials of the even terms in the general Taylor Series.

   So, using that information we can write the general form $\sum_{k=0}^{\infty}(-1)^k \frac{(\left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}$
   Notes on the construction: $(-1)^k$ forces the terms to oscillate, and since the first term was positive, we used the power k so that $(-1)^0 = 1$ rather than -1.
   The numerator term $\left(x - \frac{\pi}{2}\right)^{2k}$, the $2k$ ensures that we are only raising that term to an even power for each term of the series. So, we will only have even powers in our series.
   The denominator term $(2k)!$ ensures that we are dividing by the associated factorial for the power we are using, again, only the even terms of the series.
   Once we have the form, we can apply the ratio test and determine the convergence of this series.
   $$\lim_{k\to\infty}\left|\frac{\frac{\left(x - \frac{\pi}{2}\right)^{2(k+1)}}{(2(k+1))!}}{\frac{\left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}}\right| = \lim_{k\to\infty}\left|\frac{\left(x - \frac{\pi}{2}\right)^2}{(2k+1)(2k+2)}\right| = 0$$
   So, this series is convergent for all values of x. $R = \infty$, and $I = (-\infty, \infty)$.

■

8.2.2 Combining Power Series

Given two power series, $f(x) = \sum_k c_k x^k$ and $g(x) = \sum_k d_k x^k$, these combine just like the functions.

1. Sum and Difference: $\sum_k c_k x^k \pm \sum_k d_k x^k = \sum_k (c_k \pm d_k) x^k = f(x) \pm g(x)$
2. x^m multiplication: $x^m \sum_k c_k x^k = \sum_k c_k x^{k+m} = x^m f(x)$
3. Composition: if $h(x)$ is a monomial, bx^m, then replacing x with $h(x)$ works as composition of a function. $\sum_k c_k (h(x))^k = f(h(x))$ This one is very useful!

■ **Example 8.9** Determine the radius and interval of convergence for the combined series, given that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$
If the series converges to:

1. $\frac{x^3}{1-x}$ which is formed by multiplication by a monomial $= x^3 \sum_{k=0}^{\infty} x^k$. So, the series is the same as the general form where it converges for $|x| < 1$.
   Check endpoints: $x = -1$, $x^3 \sum_{k=0}^{\infty}(-1)^k$ diverges by A. S. T.

$x = 1$, $x^3 \sum_{k=0}^{\infty}(1)^k$ diverges by divergence test.

2. $\dfrac{1}{1-3x}$ which is formed by composition with $h(x) = 3x$. So the series is of the form $\sum_{k=0}^{\infty}(3x)^k$.
   When we apply the ratio test, $\lim_{k\to\infty}\left|\dfrac{(3x)^{k+1}}{(3x)^k}\right| = \lim_{k\to\infty}|3x| = |3x| < 1$
   $|x| < \dfrac{1}{3}$, so $R = \dfrac{1}{3}$, and we need to check the endpoints.
   $x = \dfrac{-1}{3}$: $\sum_{k=0}^{\infty}(-1)^k$ which diverges by the A. S. T.
   $x = \dfrac{1}{3}$: $\sum_{k=0}^{\infty}(1)^k$ which diverges by the divergence test.
3. $\dfrac{1}{1+2x^2}$ which is formed by composition with $h(x) = -2x^2$. So, the series is of the form $\sum_{k=0}^{\infty}\left(-2x^2\right)^k$, where $r = -2x^2$, so when we apply the ratio test, we will find that $|2x^2| < 1$ for convergence of the series.
   If we solve this algebraically, $-1 < 2x^2 < 1$, but since the function $2x^2 = |2x^2|$, and when we take a square root to solve for x, we will get both of our possible values - we only look at the right end $2x^2 < 1$, so $x^2 < 1/2$, and $-\sqrt{1/2} < x < \sqrt{1/2}$.
   So, our radius is $R = \sqrt{1/2}$.
   Check the endpoints for convergence:
   $x = -\sqrt{1/2}$: $\sum_{k=0}^{\infty}(-1)^k$ Diverges by A.S.T.
   $x = \sqrt{1/2}$: $\sum_{k=0}^{\infty}(-1)^k$ also diverges by A.S.T.
   So, our interval of convergence is $(-\sqrt{1/2}, \sqrt{1/2})$.

■

8.2.3 Differentiating and Integrating Power Series

This is one of the reasons we use a power series representation for complex functions. Polynomials are nice, so we can use the power series to approximate the derivative or integral of a much messier function.

Since all terms of our power series are continuous functions, we can make some simplifications in our computations using $\sum_k c_k(x-a)^k$

1. If we define $f(x) = \sum_k c_k(x-a)^k$, it is a continuous function on a given interval I
2. Due to the form of our power series, $f(x)$ can be differentiated or integrated term-by-term to $f'(x)$ or $\int f(x)dx$ for all points in I.

■ **Example 8.10** Given that $f(x) = \dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

Represent $g(x)$ as a series based on the form of $f(x)$ if $g(x) = \dfrac{1}{(1-x)^2}$

First, we want to determine the relationship between $f(x)$ and $g(x)$

It looks like it may be $f'(x)$, let's check: $f'(x) = \dfrac{-1}{(1-x)^2}(-1) = \dfrac{1}{(1-x)^2}$ ✓It is!

So, $g(x) = \dfrac{d}{dx}f(x) = \dfrac{d}{dx}\sum_{k=0}^{\infty} x^k = \sum_{k=1}^{\infty} kx^{k-1}$ Note: the indices shift from starting at $k = 0$ to $k = 1$ because the $k = 0$ term becomes zero when we take the derivative (also, if we try to start at $k = 0$, we will have the additional term $0 * k^{-1}$ which does not make sense).

If we encounter a series that starts with $k = 0$, but we want an equivalent series that starts at another value, we can always shift the indices (k values) to ensure the series are consistent.
So, in this example, we can then shift the indices to start at $k = 0$:

If we start with $k = 1$ and our $a_k = kx^{k-1}$, then we need to represent the same first value when $k = 0$, meaning we will add one to each k in a_k.
Starting with $k = 0$ shifts $a_k = (k+1)x^k$ instead. So, the series is then $\sum_{k=0}^{\infty}(k+1)x^k$.
Then, we can evaluate the convergence of our series for $g(x)$.
We apply the Ratio test, using $a_k = (k+1)x^k$, and $a_{k+1} = (k+2)x^{k+1}$.
$\lim_{k\to\infty}\left|\frac{(k+1)x^{k+1}}{(k+1)x^k}\right| = \lim_{k\to\infty}\left|\frac{k+2}{k+1}x\right| = |x| < 1$
So, the radius of convergence: $R = 1$.
Check the endpoints to define the interval:
When $x = -1$: $\sum_{k=0}^{\infty}(k+1)(-1)^k$, the a_k is increasing as $k \to \infty$, so the series Diverges.
When $x = 1$: $\sum_{k=0}^{\infty}(k+1)(1)^k$, $\lim_{k\to\infty}(k+1) \to \infty$ Diverges by the divergence test.
So, the interval of convergence is $(-1,1)$. ■

8.3 Taylor Series

Recall the form of our Taylor Series: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots$ (infinite terms) = $\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$
When $a = 0$, this is called the MacLaurin Series.
Note: a is a specified x value that we are using to build the approximation, so the approximation is going to match the function at that point, and possibly become less and less similar to $f(x)$ as we move away from $x = a$.

As we define series to represent functions based upon these Taylor series forms, the hardest part is recognizing the pattern to represent the derivatives generally in our series notation - but, that is also the key (just like we did with sequences).

■ **Example 8.11** Find the Taylor series of $f(x)$ centered about the given value, a. Then, find the associated radius of convergence, R, and interval of convergence, I, for your approximation.
$f(x) = \cos(x)$, $a = \frac{\pi}{2}$
First, we need to construct the Taylor series. So, we need to calculate several derivatives of $f(x)$ and construct the resulting polynomial - then, we will find a representation for it using our series notation.
$f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$
Using our derivatives at $a = \frac{\pi}{2}$, we construct the first few terms of the polynomial:
$f(x) \approx f(\pi/2) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2}(x-\pi/2)^2 + \frac{f'''(\pi/2)}{6}(x-\pi/2)^3 + \frac{f^{(4)}(\pi/2)}{24}(x-\pi/2)^4$
$= 0 - 1(x-\pi/2) + \frac{0}{2}(x-\pi/2)^2 + \frac{1}{6}(x-\pi/2)^3 + \frac{0}{24}(x-\pi/2)^4$
We can notice from this that we will only have odd terms (odd derivatives, and odd powers of $(x-\pi/2)$).
To generally notate odd values, we use the standard form $2k+1$ or $2k-1$, because multiplying an integer by 2 ensures that it is even, and then we just move one up or down to have an odd value.
Since we want our series to start with $k = 0$, we use $2k+1$ so we start with the first odd term using 1 rather than -1.
This series can be represented through $\sum_{k=0}^{\infty}(-1)^k\frac{(}{x-\pi/2)^{2k+1}}(2k+1)!$
Since our derivative values alternate between ± 1, we can represent the derivative values through

$(-1)^k$ while our first term is positive.
The denominator corresponds to the $k!$ associated with each odd derivative, so we replace $k!$ with $(2k+1)!$ to ensure consistency with the terms we wrote out above.
Finally, the $(x-a)^k$ term is consistent with the original Taylor series, but we again are only keeping odd powers, so we repleace k with $2k+1$.

Convergence of the Taylor series: we will again use the Ratio test
$a_k = \dfrac{(x-\pi/2)^{2k+1}}{(2k+1)!}$, and $a_{k+1} = \dfrac{(x-\pi/2)^{2k+3}}{(2k+3)!}$
$\lim_{k\to\infty}\left|\dfrac{\frac{(x-\pi/2)^{2k+3}}{(2k+3)!}}{\frac{(x-\pi/2)^{2k+1}}{(2k+1)!}}\right| = \lim_{k\to\infty}\left|\dfrac{x-\pi/2)^2}{(2k+3)(2k+2)}\right| \to 0 < 1$ for all x values.
So, our radius of convergence is $R=\infty$, and our Interval of convergence is $I=(-\infty,\infty)$ ■

■ **Example 8.12** Find the Taylor series of $f(x)$ centered about the given value, a. Then, find the associated radius of convergence, R, and interval of convergence, I, for your approximation.
$f(x) = \dfrac{1}{x}$, $a=1$.
We determine the derivatives of $f(x)$ to build the first several terms of our approximation.
$f'(x) = \dfrac{-1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$, $f'''(x) = \dfrac{-6}{x^4}$, $f^{(4)}(x) = \dfrac{24}{x^5}$
Evaluating each function at $a=1$: $f(1)=1$, $f'(1)=-1$, $f''(1)=2$, $f'''(1)=-6$, and $f^{(4)}(1)=24$
So, $f(x) \approx 1-1(x-1)+\dfrac{2}{2}(x-1)^2-\dfrac{6}{6}(x-1)^3+\dfrac{24}{24}(x-1)^4$
This simplifies nicely, $f(x) \approx 1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4$
So, we only need to account for the alternating sign, $(-1)^k$, and our power of $(x-1)^k$.
$f(x) = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$

Convergence of the Taylor series: we will again use the Ratio test
$a_k = (x-1)^k$, $a_{k+1} = (x-1)^{k+1}$
$\lim_{k\to\infty}\left|\dfrac{(x-1)^{k+1}}{(x-1)^k}\right| = |x-1| < 1$
So, our radius of convergence is $R=1$, check the endpoints to define the interval.
$-1 < x-1 < 1$, simplifies to $0 < x < 2$, so we check $x=0$ and $x=2$.
$x=0$: $\sum_{k=0}^{\infty}(-1)^k(-1)^k = \sum_{k=0}^{\infty}1$, which diverges by the divergence test.
$x=2$: $\sum_{k=0}^{\infty}(-1)^k(1)^k = \sum_{k=0}^{\infty}(-1)^k$, which diverges by the AST because a_k is not decreasing.
So, our interval of convergence is $(0,2)$. ■

8.3.1 Convergence of Taylor Series

Recall: Remainder $R_n(x) = f(x) - P_n(x)$

A Taylor Series of a function, $f(x)$, only converges if $\lim_{n\to\infty} R_n(x) = 0$ for all values of x in I ($\forall x \in I$).

For Taylor Series, our remainder term is defined through the form of our next term in the series:
$R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where $c \in [a,x]$ and is chosen to maximize the value of our remainder (we do this to ensure that our actual error is less than our estimated remainder).

If we look back at the previous two examples, we can evaluate their remainder terms to deter-

mine the convergence of their series.
1st example: $R_n(x) = a_{n+1} = \frac{(x-pi/2)^{2n+3}}{(2n+3)!}$

We evaluate the convergence by evaluating $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{(x-pi/2)^{2n+3}}{(2n+3)!} \to 0$ for all values of x. This series is absolutely convergent because it converges for all values of x.

2nd example: $R_n(x) = a_{n+1} = (x-1)^{n+1}$
We evaluate $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty}(x-1)^{n+1}$, which only goes to zero if $|x-1| < 1$, which is consistent with the analysis in our example. Thus, we call it consistently convergent, because it is not convergent for all x, but the values of x defined by I are convergent.

8.4 Limits, Derivatives, and Integrals of Taylor Series

In this section, we are able to use Taylor Series representations to simplify computations of limits, derivatives, and integrals of functions. This is because evaluating polynomials is relatively simple. So, we can substitute the Taylor series in for the function and evaluate.

8.4.1 Limits

For limits, we previously introduced L'Hopital's rule to evaluate indeterminate limits, but we can also get the same results using Taylor Series.

■ **Example 8.13** Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$ using the Taylor (MacLaurin) series expansion of e^x
First, we define our MacLaurin series (centered about $a = 0$, since that is our limit) for $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$

Then, we substitute this form into the limit: $\lim_{x\to 0} \frac{(1+x+\frac{x^2}{2}+\frac{x^3}{6}+...)-1}{x} = \lim_{x\to 0} \frac{x+\frac{x^2}{2}+\frac{x^3}{6}+...}{x}$
This reduces, because we are not plugging in $x = 0$, we are looking at the limit as we approach it.
$\lim_{x\to 0} = 1 + \frac{x}{2} + \frac{x^2}{6} + ... = 1$
Thus, using the Taylor series, we have shown that $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. ■

■ **Example 8.14** Evaluate $\lim_{x\to 0} \frac{\sin(2x)}{x}$ using the Taylor (MacLaurin) series expansion of $\sin(2x)$.
First, we define our MacLaurin series (centered about $a = 0$, since that is our limit) for $\sin(2x) = 2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{5!} + ...$

Then, we substitute this form into the limit: $\lim_{x\to 0} \frac{2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{5!} + ...}{x} = \lim_{x\to 0} 2 - \frac{8x^2}{6} + \frac{32x^4}{5!} + ... = 2$
Thus, using the Taylor series, we have shown that $\lim_{x\to 0} \frac{\sin(2x)}{x} = 2$. ■

8.4.2 Derivatives

As we did earlier with our power series, we can determine the derivative of a function using the derivative of its series representation.

■ **Example 8.15** Verify that $\frac{d}{dx}e^x = e^x$ using its MacLaurin series

First, we define the series for $f(x) = e^x$, which we did in a previous example $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

When we apply the derivative to this expression, $\frac{d}{dx}\left(\frac{x^k}{k!}\right) = \frac{kx^{k-1}}{k!} = \frac{x^{k-1}}{(k-1)!}$

Recall that when we apply a derivative to our power series, the indices shift by 1 (starting at $k = 1$ instead of $k = 0$ in this case)

So, $\frac{d}{dx}(e^x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$ which is the same as $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ when we shift the indices to start at $k = 0$.

Thus, $\frac{d}{dx}(e^x) = e^x$ Hooray! ■

■ **Example 8.16** Verify that $\frac{d}{dx}(\cos(x)) = -\sin(x)$ using the MacLaurin series of $\cos(x)$

First, we need to define the MacLaurin series of $\cos(x)$.

$f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$.

$f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$, $f^{(4)}(0) = 1$.

$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, from this we see that we have even powers of x and the even term factorials in the denominator.

We can ensure that we are using even integers by mulitplying k by 2. We also note that this is an alternating term, so there is a $(-1)^k$ term.

We can put these together to form the series $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$.

Now that we have a general form for our series, we can differentiate it term by term.

$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = \sum_{k=1}^{\infty}(-1)^k \frac{2kx^{2k-1}}{(2k)!} = \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k-1}}{(2k-1)!}$. When we shift the indices to start at $k = 0$, this becomes $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$.

Then, we can compare this with our general form for the MacLaurin series of $-\sin(x)$.

$g(x) = -\sin(x)$, $g'(x) = -\cos(x)$, $g''(x) = \sin(x)$, $g'''(x) = \cos(x)$, $g^{(4)}(x) = -\sin(x)$

$g(0) = 0$, $g'(0) = -1$, $g''(0) = 0$, $g'''(0) = 1$, $g^{(4)}(0) = 0$

So, $-\sin(x) \approx -x + \frac{x^3}{3!}$, from this we can see that it alternates, but the first term is negative, so $(-1)^{k+1}$ will be the alternating term. Then, we can also see that there are only odd powers of x, and our factorial. So, we need to define odd terms, starting at 1. When $k = 0$, we can arrive at 1 through $2k+1$, ensuring that we always are using an odd value.

So, $-\sin(x) = \sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$ ✓They are consistent!

Therefore, $\frac{d}{dx}(\cos(x)) = -\sin(x)$. ■

8.4.3 Integrals

One of the most powerful and helpful applications of series is using them to approximate the value of an integral we otherwise cannot evaluate. We can, again, evaluate them term-by-term.

■ **Example 8.17** Use the MacLaurin series of $f(x) = \sin(x^2)$ to evaluate $\int_0^2 \sin(x^2)dx$.

Using our properties of series, we know that a composition with a monomial results in the same series with x replaced by the monomial. So, if we use the series representation of $\sin(x)$ centered about $a = 0$ (MacLaurin series), we can just substituted x^2 in for x.

The series representation for $\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, so $\sin(x^2) = \sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!}$.

This we can then integrate term-by-term: $\int_0^2 \sin(x^2)dx = \int_0^2 \sum_{k=0}^{\infty}(-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!}dx$

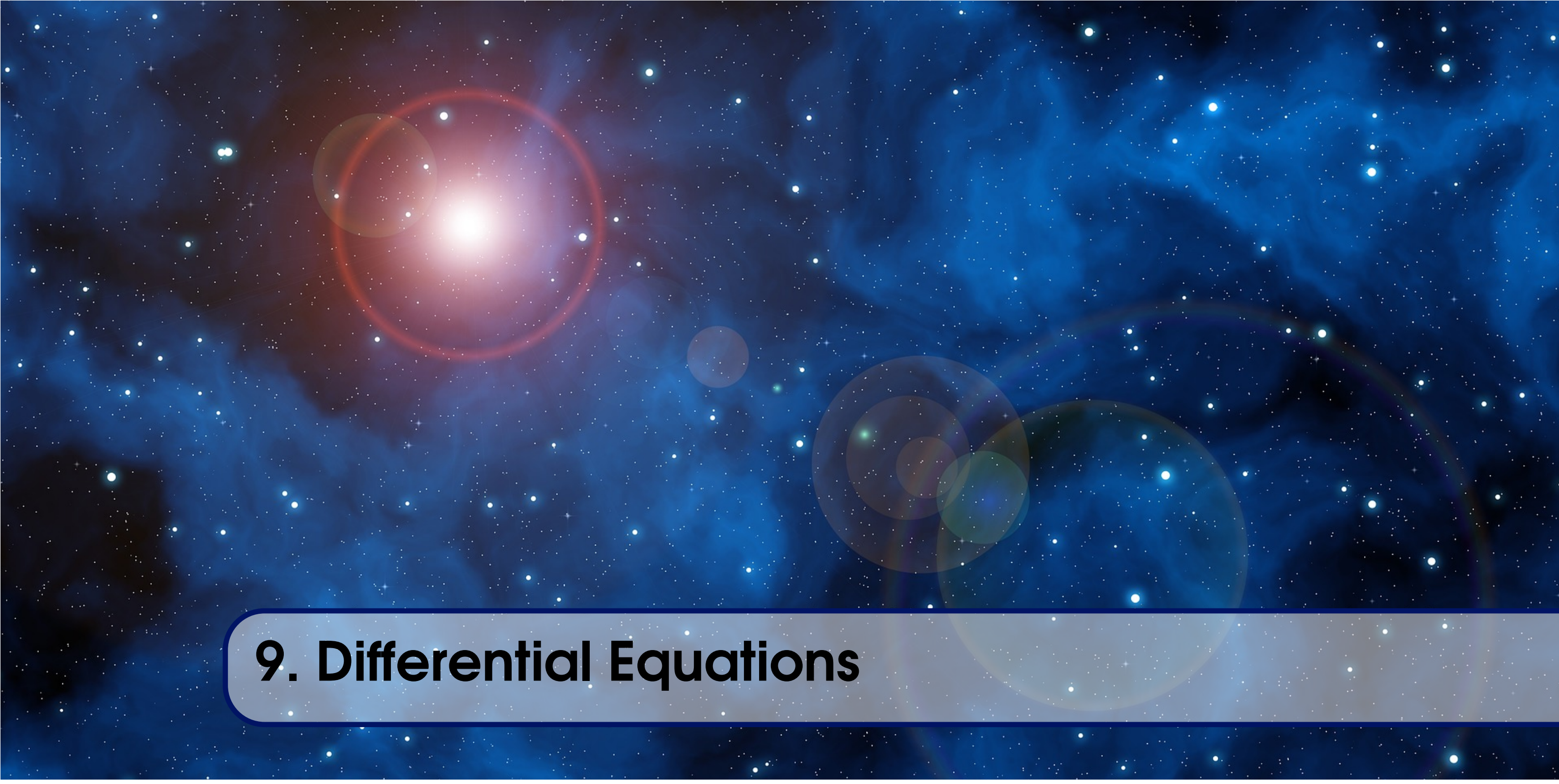
Since the terms without x are unaffected, we can pull them and the summation notation out in front of the integral.

$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^2 x^{4k+2}dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{4k+3}}{(4k+3)} |_0^2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{(2)^{4k+3}}{(4k+3)} - \frac{(0)^{4k+3}}{(4k+3)} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{2^{4k+3}}{(4k+3)}$

Since this is an alternating series, we know that we can estimate its error by the next term of the series. So, if we want an accuracy of 0.001, we need to add enough terms that the next term is less than 0.001.

$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{2^{4k+3}}{(4k+3)} \approx \frac{8}{3} - \frac{2^7}{6*7} + \frac{2^{11}}{5!*11} - \frac{2^{15}}{7!*15} + \frac{2^{19}}{9!*19} - \frac{2^{23}}{11!*23} = 2.67 - 3.0476 + 1.55151 - 0.43344 + 0.076 - 0.009 \approx 0.804$, because the next term: $\frac{2^{27}}{13!*27} \approx 0.0000025 < 0.001$.

So, we can say that $\int_0^2 \sin(x^2)dx \approx 0.804$, using the Taylor Series. ■



9. Differential Equations

9.1 Introduction to Differential Equations

Differential equations are equations that involve derivatives of a function we are solving for. In Calculus I, you even solved some basic differential equations similar to:

$\frac{dy}{dx} = 2 + \sin(x)$, where $y(0) = 3$.
You solved these by integrating the equation to solve for y, and then substituting in the value given.
$y(x) = 2x - \cos(x) + C$, $y(0) = 2 * 0 - \cos(0) + C = -1 + C = 3$, solve for $C = 4$
$y(x) = 2x - \cos(x) + 4$ satisfies the differential equation and its initial value.

This chapter is going to build off the ideas from these problems to solve more general differential equations. For example, $\frac{dy}{dx} = y$.

9.1.1 Classification of Differential Equations

We define three terms that we use to classify a given differential equation: Order, Linearity, and Homogeneity.

A differential equation's order is defined by the highest derivative in the equation. (3rd derivative = order 3, for example).

A differential equation's linearity is defined by the terms of y and its derivatives used in the differential equation. (If all the y's are linear, like $2y$, $-3y$, etc. It is linear. If there are nonlinear terms of y or its derivatives, like y^2, $\sin(y)$, etc. Then the equation is nonlinear.)

A differential equation's homogeneity is defined by the terms that do not contain y or its derivatives. (If all terms in the equation contain y or its derivatives, then the equation is homogeneous. If there are terms of x alone, or constants alone, then the equation is nonhomogeneous.)

Examples: Classify the following equations

- $\frac{dy}{dx} - y = 0$, 1st order, linear, homogeneous differential equation
- $\frac{dy}{dx} - y^2 = 2$, 1st order, nonlinear, nonhomogeneous differential equation
- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$, 2nd order, linear, homogeneous differential equation

We will start with solving 1st order differential equations.

The general form of a 1st order, linear differential equation is given by $\frac{dy}{dx} + p(x)y = q(x)$ (Note that the terms of x can be nonlinear, without affecting the linearity of the equation.)

9.1.2 Verifying Solutions to a Differential Equation

We verify solutions by taking the appropriate derivatives of the function, and substituting them into the given differential equation. We then simplify algebraically and determine if the equation is satisfied, or if it results in a contradiction.

■ **Example 9.1** Verify whether $y = Ce^{2x}$ is a solution to $y' - 2y = 0$ (Note that $\frac{dy}{dx}$ and y' are interchangeable).
To check this, we will take the first derivative of y, $y' = 2Ce^{2x}$.
Then, we substitute y and y' into the equation: $2Ce^{2x} - 2(Ce^{2x}) = 0$? Yes!
Then, $y = Ce^{2x}$ is a solution to the differential equation $y' - 2y = 0$. ■

Initial values are like the condition earlier, $y(0) = 3$, they give a point on the solution curve that we use to constrain the solution to one function that passes through that point. We also refer to this as the specific solution to the differential equation. The general solution involves an undefined constant, C. So, the general solution still satisfies the equation, but the specific solution also satisfies a given initial condition.

Since we found that $y = Ce^{2x}$ satisfied the differential equation in our example, it is the general solution for $y' - 2y = 0$. We can define a specific solution by imposing an initial value like $y(0) = 4$, which would be matched to $y(0) = C$, so we would determine that the specific solution to the differential equation and initial condition is $y = 4e^{2x}$.

■ **Example 9.2** Verify that $y = C_1\cos(2x) + C_2\sin(2x)$ is a solution to $y'' + 4y = 0$, then define the specific solution that satisfies $y(0) = 1$ and $y'(0) = 4$ (note: this is a 2nd order, linear, homogeneous differential equation).
To verify the solution, we calculate the second derivative, first $\frac{dy}{dx} = -2C_1\cos(2x) + 2C_2\cos(2x)$, second $\frac{d^2y}{dx^2} = -4C_1\cos(2x) - 4C_2\sin(2x)$.
Then, we substitute the second derivative and our original y into the given equation.
$-4C_1\cos(2x) - 4C_2\sin(2x) + 4(C_1\cos(2x) + C_2\sin(2x)) = 0$? Yes!
$y = C_1\cos(2x) + C_2\sin(2x)$ is a general solution to $y'' + 4y = 0$.

Now, we determine the specific solution.
$y(0) = C_1 * 1 + C_2 * 0 = C_1 = 1$, so $C_1 = 1$.
$y'(0) = -2C_1 * 0 + 2C_2 * 1 = 2C_2 = 4$, so $C_2 = 2$.
Therefore, the specific solution is $y(x) = \cos(2x) + 2\sin(2x)$. ■

9.1.3 Solving Differential Equations with an Antiderivative

To start solving the differential equations, we will refresh by solving first order equations of the form $y'(x) = f(x)$, which we solve using the antiderivative of $f(x)$.

■ **Example 9.3** Determine the general solution to $y'(x) = x^2 + 2$, then find the particular solution for the initial condition $y(0) = 4$.
Since $y'(x) = \frac{dy}{dx}$, when we integrate with respect to x we will determine the function $y(x)$.
$y(x) = \int y'(x)dx = \int (x^2 + 2)dx = \frac{x^3}{3} + 2x + C$

We then evaluate $y(0) = \frac{0^3}{3} + 2 * 0 + C = C = 4$

So, the specific solution is $y(x) = \frac{x^3}{3} + 2x + 4$. ■

9.1.4 Applications

Motion in a gravitational field (similar to the first few sections).

■ **Example 9.4** If you launch a ball straight up with an initial velocity $v(0) = 5m/s$, with a constant acceleration due to gravity, $a(t) = -9.8m/s^2$, and initial position $y(0) = 1m$.

1. Find the equations of motion
   First, we define $v(t) = \int a(t)dt$
   $v(t) = \int -9.8dt = -9.8t + C$
   We then use the initial value for v to define C
   $v(0) = -9.8 * 0 + C = C = 5$, so $v(t) = -9.8t + 5$
   Second, we define $y(t) = \int v(t)dt$
   $y(t) = \int (-9.8t + 5)dt = -4.9t^2 + 5t + C$
   We then use the initial value for y to define C
   $y(0) = -4.9 * 0^2 + 5 * 0 + C = C = 1$, so $y(t) = -4.9t^2 + 5t + 1$
2. At what time does the ball reach its maximum height?
   To determine the highest point, we notice that the position reaches a maximum - which means its derivative must $= 0$ at that point. (Recall: max/min problems in Calculus I)
   So, we can define the peak when $v(t) = 0$, $-9.8t + 5 = 0 \rightarrow t = \frac{5}{9.8} sec$
3. How high does the ball go?
   We can then substitute this value for t into $y(t)$ to determine the actual height (position)
   $y(5/9.8) = -4.9(5/9.8)^2 + 5(5/9.8) + 1 = -(25/19.6) + (25/9.8) + 1 = 1 + (25/19.6)m$ up.
4. When will the ball hit the ground?
   The ball hits the ground when the height $= 0$, so we solve for the time when $y(t) = 0$
   $-4.9t^2 + 5t + 1 = 0$, using the quadratic formula $t = \frac{-5 \pm \sqrt{25 - 4(-4.9)(1)}}{2(-4.9)} sec$
   Since negative time is not physically relevant to this problem, we only consider $t = \frac{5 + \sqrt{25 + 19.6}}{9.8} \approx 1.191666 sec$

■

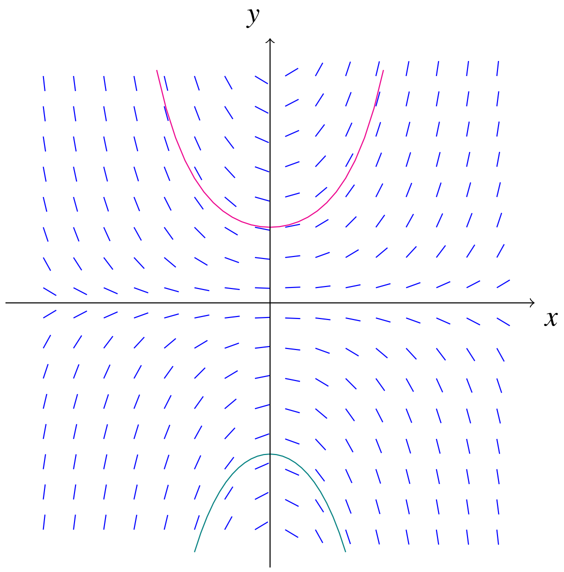
9.2 Direction Fields and Euler's Method

We use direction fields as a qualitative method to look at the form of solutions to a first order differential equation. Direction fields are a plot of the slope values at each point in space defined by $\frac{dy}{dt} = f(t,y)$.

■ **Example 9.5** Sketch the direction field for $y' = ty$
First, we want to build a table: pick values of t and y to determine the value of y', and then plot that slope at the point (t,y).

t	y	y'
0	y	0
t	0	0
1	1	1
-1	1	-1
1	2	2

Slope field of $y' = xy$.



The blue slope lines correspond to the values in the table. So, when the slope is 0, the slope line is horizontal. If the slope is 1, then the slope line is at a 45° angle, etc.
Also drawn in are some solution curves, in magenta is the solution curve for $y(0) = 1$. We can sketch this by hand by using the slope field to define the shape (any slope should be tangent to the curve at points it passes through). In teal is the solution curve for $y(0) = -2$.
The actual solution to this problem looks like $y(t) = Ce^{x^2/2}$ ■

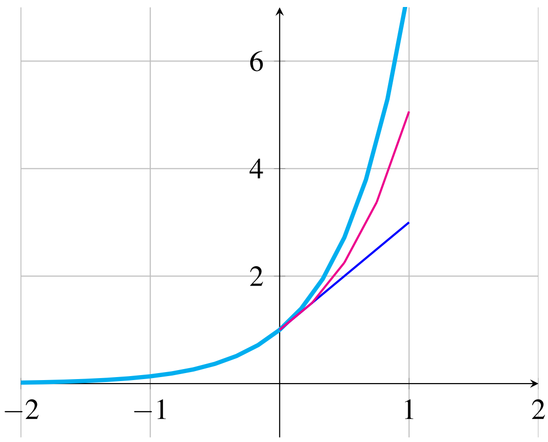
9.2.1 Euler's Method

Euler's method is a numerical solution technique that uses the slope value to approximate nearby values of $y(t)$. If we know $y' = f(t,y)$, we can use the slope and initial value to approximate the next point.

■ **Example 9.6** Approximate $y(1)$ for $y'(t) = 2y$, starting at the initial value $y(0) = 1$
Recall: the general form for a tangent line to a function $f(x)$ is given by $y - y_0 = f'(x)(x - x_0)$
So, for this problem that is modified to $y - y_0 = f(t_0, y_0)(t - t_0)$.
We will use the tangent line to estimate the value of our actual function $y(1) \approx y(0) + y'(0)\Delta t =$

$1+2*1=3$.
This is not likely to be a good approximation because taking a step of $\Delta t=1$ is rather large. So, where Euler's method becomes particularly useful is when we take smaller steps to reach the point we seek.

Let's approximate $y(1)$ using four steps instead of just one. $\Delta t=\frac{1}{4}$.
Then, we do the same calculation, but we have to complete it 4 times.
$y(1/4)\approx y(0)+y'(0)\Delta t=1+2*1/4=3/2$
Then, we use this approximation to build the next, and so forth.
$y(1/2)\approx y(1/4)+y'(1/4)\Delta t=3/2+3*1/4=9/4$
$y(3/4)\approx y(1/2)+y'(1/2)\Delta t=9/4+9/2(1/4)=27/8$
$y(1)\approx y(3/4)+y'(3/4)\Delta t=27/8+27/4(1/4)=81/16=5.0625$
The actual function is $y(t)=e^{2t}$, so $y(1)\approx 7.389$. From this, you can see that the approximation does get better with smaller steps - for better accuracy, we probably want to try even smaller steps than $1/4$.
Our absolute error for one step was $|7.389-3|=4.389$ Huge!
Our absolute error for four steps was $|7.389-5.0625|=2.3265$ Large, but much better.
Upon analyzing this method, we actually find that our error is generally cut in half for each time we double the number of steps (linear convergence).
The approximation is not going to be great, but it is handy for problems we cannot solve exactly. It also takes a lot of work to get decent accuracy.
This approximation is particularly poor because e^{2t} is a very steep function. This is better seen with a visual: see the graph below of the function, and our two approximations.



The cyan curve is the actual function, the blue line is the approximation using only one step, and the magenta curve is the approximation using 4 steps.

We are trying to approximate the value of $y(1)$ without knowing $y(t)$, and only knowing $y'(t)$. This is why we use the tangent line at each point to propagate the approximation to the curve. ■

9.3 Separable Differential Equations

We finally get to learn some solution techniques! A separable differential equation is one that can be separated so that all the terms of y are on one side, and all the terms of t are on the other side in one term (often the result of multiplication in the $f(t,y)$ function). Note: $y'=ty$ is separable, but $y'=y+t$ is NOT separable.

■ **Example 9.7** Solve $y' = ty$ for its general solution, then determine the specific solution that satisfies $y(0) = 4$.
This is separable because we can divide both sides by y, and have an equation in separated form
$\frac{y'}{y} = t$
To solve this, we rewrite it as $\frac{dy}{dt}\frac{1}{y} = t$, $\frac{dy}{y} = tdt$
From here, we can integrate both sides to get the general solution form: $\int \frac{1}{y}dy = \int tdt$
$\ln|y| = \frac{t^2}{2} + C$ (we only need one $+C$ term, since adding a constant to both sides is redundant, we add it to the t side of the equation)
$y = e^{t^2/2+C} = e^C e^{t^2/2} = ke^{t^2/2}$ where $k = e^C$.
So, the general solution to this separable equation is $y(t) = ke^{t^2/2}$.

To determine the specific solution satisfying $y(0) = 4$, we evaluate $y(0)$
$y(0) = ke^0 = k$, so $k = 4$, and $y(t) = 4e^{t^2/2}$ ■

■ **Example 9.8** Solve $y' = \frac{1}{t}$ for its general solution, then determine the specific solution that satisfies $y(1) = 2$.
This equation is already in separable form, so we rewrite $\frac{dy}{dt} = \frac{1}{t}$, and integrate both sides.
$dy = \frac{1}{t}dt$, so $\int dy = \int \frac{1}{t}dt$, and $y(t) = \ln|t| + C$
So, the general solution is $y(t) = \ln|t| + C$.

To solve for the specific solution, we evaluate $y(1)$
$y(1) = \ln|1| + C = 0 + C = C$, so $C = 2$, and the specific solution is $y(t) = \ln|t| + 2$ ■

■ **Example 9.9** Solve $y' = e^{y-t}$ for its general solution, then determine the specific solution that satisfies $y(0) = -\ln(3)$.
Using our knowlege of exponential functions, we see that this is the same as $y' = e^y e^{-t}$, so we can separate our y's and our t's.
$\frac{dy}{dt} = e^y e^{-t}$, and $\frac{dy}{e^y} = e^{-t}dt$, or $e^{-y}dy = e^{-t}dt$
Integrating both sides yields $\int e^{-y}dy = \int e^{-t}dt$, $-e^{-y} = -e^{-t} + C$
Solving algebraically for y: $e^{-y} = e^{-t} - C$, $-y = \ln(e^{-t} - C)$, $y(t) = -\ln(e^{-t} - C)$.
So, the general solution is then $y(t) = -\ln(e^{-t} - C)$.

To determine the specific solution, we evaluate $y(0)$
$y(0) = -\ln(e^0 - C) = -\ln(1 - C)$, so $1 - C = 3$, and $C = -2$.
The specific solution is then $y(t) = -\ln(e^{-t} + 2)$ ■

9.3.1 Implicit Solutions

Recall: implicit differentiation was a differentiation technique that allowed us to determine $\frac{dy}{dx}$ without isolating $y(x)$ first.

Implicit solutions use a similar concept, where we will not have to isolate $y(t)$ to define the solution.

■ **Example 9.10** Determine the implicit solution to the separable equation $\sin(y)y' = \cos(t)$

We replace y' with $\frac{dy}{dt}$, $\sin(y)dy = \cos(t)dt$
Integrate both sides $\int \sin(y)dy = \int \cos(t)dt$
$-\cos(y) = \sin(t) + C$, so $\cos(y) = -\sin(t) - C$ is an implicit solution to the original differential equation. It is an implicit solution because y is inside the function cos() ■

9.4 Special Differential Equations

In addition to separable equations, we can also solve another special form of first order differential equations: first order, linear differential equations of the form $y'(t) = ky + b$.
In this equation, we can think of $y'(t)$ as the rate of change in y, k is a constant that defines a natural growth or decay rate (e^{kt}), and b is another constant that defines some external growth or decay to the function.

This form of equation has a special form for its general solution: $y(t) = Ce^{kt} - \frac{b}{k}$. We will assume the solution is of this form, and use that to determine the specific solution.

■ **Example 9.11** Determine the general form of solution for $y'(t) = 3y - 4$, and then the specific solution that satisfies $y(0) = 1$.
Using the general form, $y(t) = Ce^{3t} + \frac{4}{3}$, let's verify that this solution works.
$y'(t) = 3Ce^{3t}$, and $3y - 4 = 3(Ce^{3t} + \frac{4}{3}) - 4 = 3Ce^{3t} + 4 - 4 = 3Ce^{3t}$ ✓
So, the general solution is $y(t) = Ce^{3t} + \frac{4}{3}$.

To find the specific solution, we need to evaluate $y(0)$
$y(0) = C + \frac{4}{3} = 1$, $C = \frac{-1}{3}$, so the specific solution is $y(t) = \frac{-1}{3}e^{3t} - \frac{4}{3}$. ■

■ **Example 9.12** Determine the general form of solution for $y'(t) - 2y = 8$, and then determine the specific solution that satisfies $y(0) = 0$.
We algebraically manipulate it into the same form as the special case: $y'(t) = 2y + 8$.
Then, using the general form, $y(t) = Ce^{2t} - \frac{8}{2} = Ce^{2t} - 4$, let's verify that the solution works.
$y'(t) = 2Ce^{2t}$, and $2y + 8 = 2(Ce^{2t} - 4) + 8 = 2Ce^{2t} - 8 + 8 = 2Ce^{2t}$ ✓
So, the general solution is $y(t) = Ce^{2t} - 4$.

To find the specific solution, we need to evaluate $y(0)$
$y(0) = C - 4 = 0$, so $C = 4$, and the specific solution is $y(t) = 4e^{2t} - 4$. ■

9.5 Modeling with Differential Equations

Differential equations are used to model a large variety of physical systems, because derivatives define rates of change. This section covers a couple models using first order differential equations.

9.5.1 Population Models

Exponential growth: $y'(t) = ry$

This is a separable equation, with general solution $y(t) = Ce^{rt}$. r defines the rate of growth or decay ($r > 0$ growth, $r < 0$ decay).

This model defines uninhibited exponential growth.

To refine this model, we need to inhibit the growth (this models reality more closely).

Logistic growth: $y'(t) = ry(1 - y/K)$, in this equation r is the same, but K is the carrying capacity of the system. It defines the maximum population supported.

We can solve this using partial fraction decomposition.

■ **Example 9.13** Solve $y'(t) = 0.2y(1 - y/300)$

$$\frac{dy}{dt} = 0.2y(1 - y/300)$$

$$\frac{dy}{y(1 - y/300)} = 0.2dt$$

Integrating both sides requires partial fraction decomposition, $\int \frac{dy}{y(1 - y/300)} = \int 0.2dt$

$$\frac{1}{y(1 - y/300)} = \frac{A}{y} + \frac{B}{1 - y/300}$$

$1 = A - \frac{Ay}{300} + By$, so $A = 1$, and $0 = \frac{-1}{300} + B$. So, $B = \frac{1}{300}$

$$\frac{1}{y(1 - y/300)} = \frac{1}{y} + \frac{1/300}{1 - y/300} = \frac{1}{y} + \frac{1}{300 - y}$$

Substituting this back into the integral, $\int \frac{1}{y} + \frac{1}{300 - y} = \int 0.2dt$

$\ln|y| - \ln|300 - y| = 0.2t + C$

Simplifying algebraically, we can solve for $y(t)$

$$\ln\left|\frac{y}{300 - y}\right| = 0.2t + C$$

$$\frac{y}{300 - y} = e^{0.2t + C} = ke^{0.2t}$$

$$y = ke^{0.2t}(300 - y)$$

$$y = 300ke^{0.2t} - kye^{0.2t}$$

$$y(1 + ke^{0.2t}) = 300ke^{0.2t}$$

$$y = \frac{300ke^{0.2t}}{1 + ke^{0.2t}}$$

Which can simplify all the way to the form you learned in algebra: $y(t) = \frac{300}{1/ke^{-0.2t} + 1}$

Define the specific solution if $y(0) = 50$

$y(0) = \frac{300k}{1 + k} = 50$, so $300k = 50 + 50k$, $250k = 50$, and $k = \frac{1}{5}$

So, the specific solution is $y(t) = \frac{300(1/5)e^{0.2t}}{1 + 1/5e^{0.2t}} = \frac{300}{5 + e^{0.2t}}$ ■